TESSELLATE PRESENTS

Scholastic Test of Excellence in Mathematical Sciences

(S.T.E.M.S.)

Title Sponsor

Computer Science Sample Paper

College Category

2019

Message from The S.T.E.M.S. Team

Dear applicant,

Greetings from team Tessellate. This is the first edition of Scholastic Test of Excellence in Mathematical Sciences. All that we have to say before the exam is: DO NOT BE NERVOUS! The questions can be too hard for some of you, and too easy for the rest. Remember, the aim should not be to solve every question completely. Rather, try to attempt the questions, and write whatever you are thinking, even if it is a couple of lines. If you think that you are not being able to solve the questions, and they are too hard, well then they will be hard for everyone and the ranking will be relative. We are NOT looking for exactly correct answers for all the questions! Rather, we would like to see what you are thinking. We shall be as lenient as possible with our grading, and try to give as much part marks as possible in every subjective question.

Best wishes! Rock the exam.

Rules and Regulations

Submission of Answer sheets

1. The digital copy of the answer sheet has to be mailed to `xxxxx@yyyy.zzz`. The submissions should be sent strictly within the time limit.
2. You are allowed to scan/take pictures of your answer sheet for submission. You may use TeX if you wish to submit a soft copy of the solution as a pdf file.
3. Ensure that your submitted copy is clear and readable. Illegible submissions such as bad handwriting and unclear photographs will be disqualified.
4. Send the entire answer sheet in one single mail. Repeated mails of solutions will not be accepted.
5. Email back the answer sheet from the same email id you registered from (this is the same id that you received the question paper in).
6. Do NOT write your name or college/location in your answer sheet. Write your registration ID if you have it.

Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question is worth 10 marks each.
   - If you mark exactly the set of correct answers as correct, you get 10 marks
   - Otherwise, you
     - gain 2 points for every answer you marked and is actually correct
     - and loose 1 point for every answer you marked but is actually incorrect.
3. You are not required to show your work for the objective part of the paper.
4. For Objective questions having choice, each problem has at least one correct answer: *however, there might be many.* Choose ALL that apply, unless indicated otherwise.
5. For Problems 1-8, write your answers in the form of a tabular column where the rows denote the question number in order and the columns options *(a)* to *(e)* and tick the appropriate grids. For questions with no option (e), ignore the column with option e. For instance if you want to mark the answers $(a), (d)$ for question 1, and option

$(c), (d)$ for question 7, then this is how the corresponding row would look like:

	(a)	(b)	(c)	(d)
(1)	✓			✓
			⋮	
(7)			✓	✓

6. Each subjective problem is worth 20 marks each.
7. For the subjective questions you need to formally show your reasoning. Marks will be awarded for partially correct solutions.
8. The subjective problems will be graded only if you score above a certain cut-off (which will be decided later) in the objective section of the paper.
9. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. You are not allowed to post/discuss the problems in any online forum within the exam time.
4. For any querries regarding the paper, please email `xxxxx@yyyy.zzz`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `xxxxx@yyyy.zzz` .

Best Wishes for your Exam!

Objective Questions

Problem 1. *Mr. Sen buys and sells potatoes to make a profit. Due to inventory restrictions, he always has to buy and sell 1 kg of potatoes and cannot store more than 1 kg either.*

This is the price of a kilogram of potatoes throughout the next 10 days, which he has found out from his private sources. He does not own any potato at the beginning.

Day	1	2	3	4	5	6	7	8	9	10
Price	11	7	10	9	13	14	10	15	12	10

There is also a fee of 1 every time he either buys or sells a kilogram of potatoes.

What is the maximum profit that he can make?

(a) 6

(b) 7

(c) 8

(d) 9

Problem 2. *Suppose* L_1 *and* L_2 *are two languages over some finite alphabet. Suppose* $L_1 \cdot L_2$ *is recursive.*

Then, which of the following should also be recursive?

(a) L_1

(b) L_2

(c) $L_2 \cdot L_1$

(d) None of the above

Problem 3. *A connected graph has at least 100 vertices and 500 edges. Suppose the minimum number of distinct cycles that it must have is k. Then, which of the following is the closest to k?*

(a) 100

(b) 200

(c) 300

(d) 400

Problem 4. *Let x and y be strings and let L be any language over some fixed finite alphabet. We say that x and y are* ***distinguishable*** *by L if some string z exists whereby exactly one of the strings xz and yz is a member of L; otherwise, for every string z, we have $xz \in L$ whenever $yz \in L$ and we say that x and y are* ***indistinguishable*** *by L. If x and y are indistinguishable by L, we write $x \equiv_L y$.*

Given any regular language L which of the following languages always has the same $\equiv_L$ relation as L?

(a) L^*

(b) L', where $L' \subseteq L$

(c) L^c, complement of L

(d) L', where $L \subseteq L'$

Problem 5. *Consider the following functions of n:*

$$n^2, n!, 2^n, \binom{2n}{n}, n^{log^2(n)}$$

Which of the following options have them sorted in the correct order of growth?

(a) $n^2 \leq n^{log^2(n)} \leq 2^n \leq n! \leq \binom{2n}{n}$

(b) $n^{log^2(n)} \leq n^2 \leq 2^n \leq n! \leq \binom{2n}{n}$

(c) $n^2 \leq n^{log^2(n)} \leq 2^n \leq \binom{2n}{n} \leq n!$

(d) $n^2 \leq n^{log^2(n)} \leq \binom{2n}{n} \leq 2^n \leq n!$

Problem 6. *A linear order is a set with a relation $\leq$ which is antisymmetric, transitive and total.*

There are two linear orders $\mathfrak{A}$ and $\mathfrak{B}$ which contain m and n elements respectively, with $m \neq n$.

Mr. Spoiler and Mr. Duplicator are playing a game where Mr. Duplicator wants to fool Mr. Spoiler into believing that the linear orders are isomorphic in the sense explained below. The game proceeds as follows for k rounds.

1. *The first player, Mr. Spoiler, picks either a member a_1 of $\mathfrak{A}$ or a member b_1 of $\mathfrak{B}$.*
2. *If Mr. Spoiler picked a member of $\mathfrak{A}$, Mr. Duplicator responds by picking a member $_1 \in \mathfrak{B}$; otherwise Mr. Duplicator picks a member $1 \in \mathfrak{A}$.*
3. *Again Mr. Spoiler picks either $a_2 \in \mathfrak{A}$ or $b_2 \in \mathfrak{B}$, and Mr. Duplicator responds by picking an element from the linear order other than the one from which the duplicator picked.*
4. *This continues for k rounds, and sequences $a_1, a_2, a_3 \cdots a_k$ and $b_1, b_2, b_3 \cdots b_k$ are chosen.*

Now, Mr. Duplicator is said to have won the game if for every i, j, $a_i \leq a_j$ if and only if $b_i \leq b_j$. Otherwise Mr. Spoiler has won.

Suppose now that $m, n \geq \alpha$ for some α. How should α be related to k such that Mr. Duplicator has a winning strategy?

(a) $\alpha = \mathcal{O}(k)$

(b) $\alpha = \mathcal{O}(k^2)$

(c) $\alpha = \mathcal{O}(2^k)$

(d) Mr. Duplicator can never win

Problem 7. *Which of the following statements are correct?*

(a) L is regular, given L^2 is regular.

(b) Given L is regular, $root(L)$ is regular, where $root(L) = \{w \mid ww \in L\}$

(c) Given L is regular, $square(L)$ is regular, where $square(L) = \{ww \mid w \in L\}$

(d) Given L is regular, $Pal(L)$ is regular, given $Pal(L) = \{ww^r \mid w \in L\}$

Problem 8. *Let $\mathcal{P}$ be a family of sets. We define the set Φ of formulae to be the smallest set of formulae that contains elements of $\mathcal{P}$ and is closed under implication $\to$. For example, if $P, Q, R \in \mathcal{P}$, then $P, P \to P, P \to Q, (P \to Q) \to Q, P \to (Q \to Q), ((P \to P) \to P) \to P, P \to Q \to R$ are formulae. If brackets are omitted, they are assumed to associate to the right. For example, $P \to P \to P$ stands for $P \to (P \to P)$.*

We intuitively pretend that P, Q, R are sets, and $P \to Q$ is the set of functions from P to Q, $P \to (Q \to Q)$ is the set of functions from P to the (set of functions from Q to Q), etc.

Mr. Prover and Mr. Skeptic are playing a game that goes like this:

- ***Prover:*** *I assert that there is an element in $((P \to P) \to (Q \to R \to Q) \to S) \to S$.*
- ***Skeptic:*** *Really? I give you an element e_1 in $((P \to P) \to (Q \to R \to Q) \to S)$. Can you give me an element in S?*
- ***Prover:*** *Of course, I will assert that there are elements $e_2 \in P \to P$ and $e_3 \in (Q \to R \to Q)$. Then I apply e_1 to e_2 and then apply the result to e_3. That should be an element in S.*
- ***Skeptic:*** *Okay, you made two assertions. I doubt if the second assertion is true. Suppose I give you an element $e_4 \in Q$ and an element $e_5 \in R$, can you give me an element in Q?*
- ***Prover:*** *I can just give you back e_5.*

More formally, the game proceeds like this:

1. *In the first step, the prover asserts a formula $\alpha_1 \to \alpha_2 \to \cdots \to p$ where $p \in \mathcal{P}$ but $\alpha_1, \alpha_2, \cdots \alpha_n$ maybe more complex formulae.*
2. *The skeptic offers elements $e_1 \in \alpha_1$, $e_2 \in \alpha_2$, etc and asks the prover to produce an element in p.*
3. *The prover can introduce further assertions that $f_1 \in \beta_1$, $f_2 \in \beta_2$, etc where β_1, β_2, etc are formulae. Then, he has to use $f_1, f_2, \cdots f_n$ along with $e_1, e_2, \cdots e_r$ in a way that he wishes to produce an element of p.*
4. *The skeptic can choose to challenge one of the assertions made by the prover, i.e, he picks some β_i that the prover mentioned in the last step. If $\beta_i = \gamma_1 \to \gamma_2 \cdots r$ where $r \in \mathcal{P}$, the prover again offers elements in each of $\gamma_1, \gamma_2, \cdots \gamma_m$ and asks the prover to produce something in r.*
5. *The game continues this way. The prover makes some assertions and uses these assertions along with any of the previous offers to produce the intended element. And then the skeptic again challenges one of his assertions along with the relevant offers, and so on.*

The prover wins the game if he has not introduced any new assertion and hence the skeptic cannot respond. For example, in the example play above, the prover wins. If the prover cannot respond to a challenge, or the game continues indefinitely, the skeptic is said to have won.

Which of the following are true?

(a) *The game can sometime continue indefinitely*

(b) *The prover wins on exactly all those formulae which are tautologies when the formulae is interpreted as a formula in classical logic with* $\to$ *as implication.*

(c) *If the prover started with the formula* $(P \to Q) \to P) \to P$*, then the prover wins*

(d) *If the prover started with the formula* $(P \to Q \to R) \to (P \to Q) \to P \to R$*, then the prover wins*

Subjective Questions

Problem 9. *Show that if a graph has n vertices and m edges with $2m \geq n^{3/2}$, it has a cycle of length ≤ 4*

Problem 10. *Consider a town with N people. We need to form clubs with the following properties:*

- *Each club contains even number of people.*
- *The number of common people among any 2 distinct clubs is even.*

Note that a person can join any number of clubs.

Show that the maximum number of clubs that can be formed is $2^{\lfloor N/2 \rfloor}$.

Problem 11. *Suppose G is a graph. Let M be a maximal matching in G with the least possible cardinality in G. Suppose the cardinality of M is k.*

Find a polynomial time algorithm that finds a maximal matching N in G where the cardinality of N is not more than $2k$

Problem 12. *Alice has a secret number $S < 2^N$ (with $N \leq 32$). Now, you have joined the evil forces with Eve and so you have to help Eve crack Alice's secret number, i.e., find the value of S (with high probability).*

All that Eve has access to is a system with hidden implementation to a special function `noisy_xor_with(x)`*. But the system has some restrictions and so you can only make 50,000 calls to the* `noisy_xor_with(x)` *function.*

Everytime `noisy_xor_with(x)` *is called with argument x, the system chooses a random bitstring u of length N (such that each bit is sampled independently with a probability of 0.1 of getting a 1), and then computes $u \oplus x \oplus S$ and finally returns the number of 1's in $u \oplus x \oplus S$.*

Outline an algorithm that you can use to obtain the value of S. Analyze the probability with which your algorithm finds the correct value of S. (The higher the probability, the better)

Problem 13. *Propositional logical formulae over propositional atoms* $\mathcal{P}$ *are defined in the usual way: they are either atoms, or boolean combinations of smaller formulae. A valuation* $v : \mathcal{P} \to \top, \bot$ *is a function that assigns a truth value to each propositional variable. Given such a valuation, we use standard truth table semantics to assign truth values to formulae.*

We say that a valuation $v \models \phi$ *if* v *is a satisfying assignment for* ϕ. *And if* X *is a set of formulae, we write* $v \models X$ *to say that* v *satisfies all formulae in* X.

Now, we say $X \models \phi$ *(read* ϕ *is a logical consequence of* X*) if whenever a valuation satisfies* X, *it also satisfies* ϕ. *We say that a set* X *is dependent if there exists* $\alpha \in X$ *such that* $X \setminus \{\alpha\} \models \alpha$. *For example, consider the sets* $X = \{p \wedge q, p\}$ *and* $Y = \{p, q\}$. X *is dependent since* p *is a logical consequence of the other members of the set. However,* Y *is independent since no member of the set can be deduced as a logical consequence of others.*

We say that a set X *is equivalent to a set* Y *if for every formula* α, $X \models \alpha$ *iff* $Y \models \alpha$. *For example, the sets* $\{p \wedge q\}$ *and* $\{p, q\}$ *are equivalent. However, the sets* $\{p \vee q\}$ *and* $\{p, q\}$ *are not.*

Find a set which has no independent equivalent subset. Can such a set be finite?