**TESSELLATE PRESENTS**

# Scholastic Test of Excellence in Mathematical Sciences

# (S.T.E.M.S.)

Title Sponsor

BRILLIANT

Math and science done right

## Computer Science Exam

## College Category

January 13, 2018

# Message from The S.T.E.M.S. Team

Dear applicant,

Greetings from team Tessellate. This is the first edition of Scholastic Test of Excellence in Mathematical Sciences. All that we have to say before the exam is: DO NOT BE NERVOUS! The questions can be too hard for some of you, and too easy for the rest. Remember, the aim should not be to solve every question completely. Rather, try to attempt the questions, and write whatever you are thinking, even if it is a couple of lines. If you think that you are not being able to solve the questions, and they are too hard, well then they will be hard for everyone and the ranking will be relative. We are NOT looking for exactly correct answers for all the questions! Rather, we would like to see what you are thinking. We shall be as lenient as possible with our grading, and try to give as much part marks as possible in every subjective question.

Best wishes! Rock the exam.

# Rules and Regulations

## Submission of Answer sheets

1. The digital copy of the answer sheet has to be mailed to `stems.cscol@gmail.com`. The submissions should be sent strictly within the time limit.
2. You are allowed to scan/take pictures of your answer sheet for submission. You may use TeX if you wish to submit a soft copy of the solution as a pdf file.
3. Ensure that your submitted copy is clear and readable. Illegible submissions such as bad handwriting and unclear photographs will be disqualified.
4. Send the entire answer sheet in one single mail. Repeated mails of solutions will not be accepted.
5. Email back the answer sheet from the same email id you registered from (this is the same id that you received the question paper in).
6. Do NOT write your name or college/location in your answer sheet. Write your registration ID if you have it.

## Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question is worth 3 marks each.
3. You are not required to show your work for the objective part of the paper.
4. For Objective questions having choice, each problem has at least one correct answer: *however, there might be many.* Choose ALL that apply, unless indicated otherwise.
5. There are 4 or 5 options for problems 1-12 and marks are awarded if and only if EXACTLY the right set of options are picked.
6. For Problems 1-12, write your answers in the form of a tabular column where the rows denote the question number in order and the columns options *(a)* to *(e)* and tick the appropriate grids. For questions with no option $(e)$, ignore the column with option $e$. For instance if you want to mark the answers $(a), (d), (e)$ for question 1, and option $(c), (d)$ for question 12, then this is how the corresponding row would look

like:

| | (a) | (b) | (c) | (d) | (e) |
|---|---|---|---|---|---|
| (1) | ✓ | | | ✓ | ✓ |
| | | | ⋮ | | |
| (12) | | | ✓ | ✓ | |

7. Problems 13 and 14 are objective problems and DO NOT require detailed solutions. Just write the answer asked for by the question.
8. Each subjective problem is worth 10 marks each.
9. For the subjective questions you need to formally show your reasoning. Marks will be awarded for partially correct solutions.
10. The subjective problems will be graded only if you score above a certain cut-off (which will be decided later) in the objective section of the paper.
11. There is no negative marking.
12. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.

## Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. You are not allowed to post/discuss the problems in any online forum within the exam time.
4. For any querries regarding the paper, please email `stems.college.cs@gmail.com`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `stems.tessellate@gmail.com` .

Best Wishes for your Exam!

# Objective Questions

**Problem 1.** *Which of the following is true about* $2^{4n^3+7}$. *It is*

(a) $O(100n^{3(\log n)^4})$

(b) $\Omega(42^{42n^2})$

(c) $O((\sqrt[3]{n})^{(n\log n)^2+764})$

(d) $\Omega(\sqrt{f(n)}^{f(n)})$ *where* $f(n)$ *is defined recursively as follows:* $f(n) = 5\log n + f(n/2)$, *and* $f(1) = 1$ *and*

(e) $O(g(n))$, *where* $g(n)$ *is defined recursively as follows:* $g(n) = g(n-1) + g(n-2)$, *and* $g(0) = 0$ *and* $g(1) = 1$

**Problem 2.** *Which of the following languages are decidable: (*$\langle M\rangle$ *denotes an encoding of* $M$*)*

(a) $\{\langle M_1, M_2\rangle \mid M_1$ *and* $M_2$ *are DFAs and* $L(M_1) = L(M_2)\}$

(b) $\{\langle M_1, M_2\rangle \mid M_1$ *is a DFA,* $M_2$ *is a turing machine such that* $L(M_1) = L(M_2)\}$

(c) $\{\langle M\rangle \mid M$ *doesn't run for more than* $7n$ *steps for any input* $w$, *where* $|w| = n\}$

(d) $\{\langle M\rangle \mid M$ *doesn't run for more than* $2187$ *steps for any input* $w$, *where* $|w| = n\}$

(e) $\{\langle M\rangle \mid$ *where* $M$ *is the description of a Non-deterministic Turing machine*$\}$

**Problem 3.** *Which of the following problems are NP-hard (Assume P≠NP) where the input is assumed to be* $(G, k)$ *where* $G = (V, E)$ *is an encoding of a graph and* $k$, *an integer.*

(a) *Checking if* $G$ *is a tree and* $G$ *has a vertex cover of size at most* $k$.

(b) *Finding if there are* $k$ *cycles in* $G$ $C_1, \ldots, C_k$ *such that* $C_i$ *and* $C_j$ *don't share an edge.*

(c) *Checking if the is a subset* $U \subset P$ *such that* $|U| \leq k$ *such that the graph restricted to* $V \setminus U$ *is bipartite.*

(d) *Checking if a graph is planar and can be coloring using 10 colors*

(e) *All of the above*

**Problem 4.** *Which of the following denotes* $T_n$ *where* $T_0 = 1$ *and*

$$T_n = \sum_{i=0}^{n-1} T_i T_{n-i}$$

(a) *The number of 'valid bracketings' with* $n$ *open brackets. A valid bracketing is one where every open bracket has a matching closed bracketing.* $(())()$ *is a valid bracketing, whereas* $())(()$ *is not.*

(b) *The number of ways of ways of dividing a convex polygon with* $n$ *vertices into triangles*

(c) *The number of rooted binary trees with* $n$ *internal nodes.*

(d) *The number of ways of choosing* $n$ *objects from* $2n$ *distinct objects*

(e) *None of the above*

**Problem 5.** *Consider the following augmentation to the model of push down Automata. In the original Push Down Automata, recall that the head of the automata moves in exactly one direction. Here, we allow the head to move both towards the left and the right. The automaton can also recognize if the head is at the beginning or the end. Similar to the previous model, we have a stack in which we can push, pop and check the top most element of the stack. Which of the following languages can be accepted by the above modification of the Push Down Automata.(We use* $\Sigma$ *to denote alphabet, where* $|\Sigma| > 1$*)*

(a) $\{a^i b^i c^i \mid i \in \mathbb{N}\}$ *, where* $a, b, c \in \Sigma$

(b) $\{w.w \mid w \in \Sigma^*\}$

(c) $\{w.w^r \mid w \in \Sigma^*\}$*, where* $w^r$ *denotes the reverse of the string* $w$

(d) $\{a^n b^m \mid m = n^2\}$*, where* $a, b \in \Sigma$

(e) $\{w \mid$*number of 'a's occurring in* $w$ *equals number of 'b's* $\}$

**Problem 6.** *Which of the following are true (Standard notations apply)?*

(a) $NTIME(n^{1000}) \subseteq DSPACE(n^{1000})$

(b) $DTIME(n) = NTIME(n^2)$

(c) $RP = NEXP$

(d) $NSPACE(n^{\log n}) \subseteq DSPACE(n^{\log \log n})$

(e) $NSPACE(n^2) \subsetneq DSPACE(n^8)$

**Problem 7.** *Which of the following is true about* **Algorithm 1**

**Algorithm 1**

```
procedure ALGORITHM:((G = (V, E), w))
    Let E = {e_1, ..., e_m} where w(e_1) ≥ w(e_2) ≥ ... w(e_m)
    for i in [1,...,m] do
        If deleting e_i doesn't disconnect the graph:
            Delete e_i from graph G
    return Remaining edges of G
```

(a) *The algorithm returns the minimum weight cycle in the graph G*

(b) *The algorithm returns the maximum spanning forest of the graph G*

(c) *When the weights of the graph are distinct, and if input graph G is connected, then the algorithm finds the minimum spanning tree of G*

(d) *If input graph G is connected, then the algorithm finds a minimum spanning tree of G*

(e) *If input graph G is connected, then the algorithm finds a maximum spanning tree of G*

**Problem 8.** *Let $G = (V, E)$ be a weighted graph with weights given by the function $w : E \to \mathbb{Z}$. The smallest weight path between $u$ and $v$ is $d$. There is only one path from $u$ to $v$ which has weight $d$ and that path has $k$ edges. Define a modified weight function $w'$ such that $w'(e) = w(e) + c$, where $c$ is a constant. Then, which of the following is true with weight now being $w'$?*

(a) *Shortest weight path between $u$ and $v$ is $d - c$.*

(b) *Shortest weight path between $u$ and $v$ is $d + c$.*

(c) *Shortest weight path between $u$ and $v$ is $d + (k * c)$.*

(d) *Shortest weight path between $u$ and $v$ is $d$.*

(e) *None of the above*

**Problem 9.** *A real number $x \in [0,1]$ is said to be computable if there exists a computable function $f : \mathbb{N} \to \{0,1\}^*$ such that given any $b \in \mathbb{N}$, $f(b)$ is the first $b$ significant bits of $x$. Informally, $x$ is computable if a turing machine can approximate it to arbitrary accuracy.*

*Now, suppose $a_n$ is a sequence of real numbers in the interval $[0,1]$ such that each element in the sequence is computable. Furthermore, suppose that $a_n$ converges to $a$. What can you say about $a$?*

*(a) $a$ is not necessarily in $[0,1]$*

*(b) $a$ is never computable*

*(c) $a$ is always computable (if you check this option, also check the next option)*

*(d) $a$ is sometimes computable*

**Problem 10.** *In a party, there are seven logicians. Each of them are dancing, or not dancing. To describe properties of the party, they consider the First Order Language with the unary predicate `Dancing`. Consider the following sentence:*

$$\exists x.[\texttt{Dancing}(x) \to \forall y.\texttt{Dancing}(y)]$$

*(a) This sentence is false in all models*

*(b) This sentence is true in all models (if you check this option, also check the next option)*

*(c) This sentence is true in some models*

*(d) This sentence is syntactically ill-formed*

**Problem 11.** *Suppose $\Sigma$ is a finite alphabet with symbols 0 and 1. Consider the two languages:*

$$L_1 = \{xy | x \in \Sigma^*, y \in \Sigma^*, \#0(x) = \#1(y)\}$$
$$L_2 = \{x\$y | x \in \Sigma^*, y \in \Sigma^*, \#0(x) = \#1(y)\}$$

*Here, \$ is a symbol other than 0 and 1. $\#0(.)$ and $\#1(.)$ represent the number of 0's and 1's respectively in the strings . Which of these two languages are regular?*

*(a) Neither of them*

*(b) Only $L_1$*

*(c) Only $L_2$*

*(d) Both of them*

**Problem 12.** *Suppose $G$ is a bipartite graph. Suppose the maximum matching on $G$ has $n$ edges. When does the minimum vertex cover also have $n$ vertices? (Select only one option)*

(a) *Always*

(b) *Whenever the maximum matching is also a perfect matching*

(c) *Whenever there are an equal number of vertices in either bipartition*

(d) *Never*

**Problem 13.** *A subset $L \subseteq \{0,1\}^n$ is closed under permutation if for every word $w = a_1.a_2 \ldots a_n$, with $w \in L$, $a_i \in \{0,1\}$, then we also have $a_{\sigma(1)}.a_{\sigma(2)} \ldots a_{\sigma(n)} \in L$, where $\sigma$ denotes a permutation. For fixed $n$, give a closed form for the number of subsets $L$ of $\{0,1\}^n$ which are closed under permutation.*

**Problem 14.** *In the city of Metropolis, the population of each house in a block is represented as an $10 \times 10$ grid as shown below. A strange virus created by Lex Luther works the following way: If a block has population $n$, then every block in the same row or column with population $m$ such that the sum of the digits of $m$ is strictly smaller than the sum of the digits of $n$, will be infected in the next step. i.e, if the population of the block is 78 ($n = 78$), the sum of the digits is $7 + 8 = 15$. And if the population of any house in the same row or column has sum of the digits $< 15$ ($m = 11, 45, 76$etc.,), then it will also be infected. Lex Luthor now plans to infect the city by introducing the virus in the least possible number of houses. Superman comes to know of Lex Luthor's plan and wants to prevent it. Help Superman defeat Lex Luthor by reporting the number of houses Lex Luthor had selected along with the population of the house with the maximum digit sum among them.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 60 | 31 | 10 | 45 | 31 | 67 | 54 | 90 | 76 |
| 13 | 03 | 37 | 24 | 63 | 07 | 25 | 19 | 88 | 32 |
| 51 | 45 | 15 | 59 | 78 | 73 | 42 | 87 | 06 | 01 |
| 71 | 16 | 36 | 31 | 01 | 46 | 66 | 62 | 32 | 77 |
| 98 | 93 | 40 | 63 | 73 | 71 | 57 | 30 | 90 | 38 |
| 74 | 97 | 08 | 06 | 92 | 65 | 26 | 74 | 10 | 34 |
| 45 | 43 | 29 | 03 | 64 | 13 | 49 | 73 | 84 | 83 |
| 69 | 43 | 05 | 54 | 90 | 14 | 26 | 25 | 11 | 86 |
| 47 | 97 | 33 | 58 | 70 | 69 | 56 | 31 | 93 | 42 |
| 79 | 04 | 16 | 16 | 03 | 78 | 40 | 90 | 27 | 52 |

# Subjective Questions

**Problem 15.** *Given a directed graph $G$ with $n$ vertices and a special vertex $u$ among the vertices of $G$. We call a vertex $v$ 'interesting' if there is a path from $v$ to a vertex $w$ such that there is a cycle containing the vertices $w$ and $u$. Given an $O(n)$ time algorithm which takes as input $G$ along with $u$ returns all the interesting vertices.*

**Problem 16.** *Consider a graph $G = (V, E)$ with $n$ vertices and $m$ edges. Let $w$ be a weight function such that $w : E \to \mathbb{Q}$ which assigns distinct weights to each edge. Show that there is always a walk of length at least $2m/n$ such that the weight of the edges along this walk forms a strictly decreasing sequence. A walk is a sequence $v_0, e_1, v_1, \ldots, v_k$ of graph vertices $v_i$ and graph edges $e_i$ are distinct and are also such that for $1 \leq i \leq k$, the edge $e_i$ has endpoints $v_{i-1}$ and $v_i$. The length of a walk is its number of edges.*

**Problem 17.** *You are given a $n \times 3$ grid. Using the $3 \times 1$ tiles and $2 \times 2$ tiles we want to tile the $n \times 3$ grid completely. Let $f(n)$ denote the number of ways you can you cover the grid using just these tiles. Write an algorithm that would take as input $n$, and output $f(n)$*

**Problem 18.** *Aalok has gifted Eku a toy. This toy is in the form of an $n \times n$ grid, and each cell either has a euro or doesn't have a euro. This toy is constrained such that any two rows can be swapped any number of times but individual blocks in the grid cannot be moved. To be able to access all the euros, Eku needs to swap these rows such that all the elements in the main diagonal contains euros and the other cells might or might not. (The main diagonal consists of elements of row $i$ column $i$, for each $i \in \{1, \ldots, n\}$) . Given the initial configuration of the euros, help Eku come up with an algorithm to find out if he will ever be able to get all the euros. Below is an example of one row swap in a $3 \times 3$ grid, which makes all the elements of the main diagonal contain euros.*

$$\begin{matrix} € & € & O \\ O & O & € \\ O & € & O \end{matrix} \xrightarrow{\textit{Swap Row 2 and 3}} \begin{matrix} € & € & O \\ O & € & O \\ O & O & € \end{matrix}$$