TESSELLATE PRESENTS

Scholastic Test of Excellence in Mathematical Sciences

(S.T.E.M.S.)

Title Sponsor

Mathematics Exam

College Category

January 14, 2018

Message from The S.T.E.M.S. Team

Dear applicant,

Greetings from team S.T.E.M.S. This is the first edition of Scholastic Test of Excellence in Mathematical Sciences. All that we have to say before the exam is: DO NOT BE NERVOUS! The questions can be too hard for some of you, and too easy for the rest. Remember, the aim should not be to solve every question completely. Rather, try to attempt the questions, and write whatever you are thinking, even if it is a couple of lines. If you think that you are not being able to solve the questions, and they are too hard, well then they will be hard for everyone and the ranking will be relative. We are NOT looking for exactly correct answers for all the questions! Rather, we would like to see what you are thinking. We shall be as lenient as possible with our grading, and try to give as much part marks as possible in every subjective question.

Best wishes! Rock the exam.

Rules and Regulations

Submission of Answer sheets

1. The digital copy of the answer sheet has to be mailed to `stems.mathcol@gmail.com`. The submissions should be sent strictly within the time limit.
2. You are allowed to scan/take pictures of your answer sheet for submission. You may use TeX if you wish to submit a soft copy of the solution as a pdf file.
3. Ensure that your submitted copy is clear and readable. Illegible submissions such as bad handwriting and unclear photographs will be disqualified.
4. Send the entire answer sheet in one single mail. Repeated mails of solutions will not be accepted.
5. Email back the answer sheet from the same email id you registered from (this is the same id that you received the question paper in).
6. Mention which degree you are pursuing and year. Do NOT write your name or college/location in your answer sheet. Write your registration ID if you have it.

Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question is worth **1 point** and there is no negative marking.
3. You are not required to show your work for the objective part of the paper.
4. Each subjective problem is worth **20 points**.
5. For getting full credit in the subjective questions you need to give the detailed solutions. However, credit will also be awarded for partially correct solutions.
6. There is no negative marking in the subjective section as well.
7. **The subjective part will be graded only if you score above a certain cut-off (which will be decide later) in the objective section of the paper. However, for the final score, your total score (subjective + objective) will be taken into consideration.**

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.
4. You are not allowed to post/discuss the problems in any online forum within the exam time.
5. For any querries regarding the paper, please email `stems.college.math@gmail.com`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `stems.tessellate@gmail.com` .

Notation

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ are the set of integers, rationals, positive integers, real numbers and positive real numbers respectively.
2. $\sum\limits_{\text{cyc}}$ denotes cyclic sum, the definition of cyclic sums is given at `https://artofproblemsolving.com/wiki/index.php?title=Cyclic_sum`
3. $a \mid b$ for any two integers a, b with $a \neq 0$ means that a is a factor of b.

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Objective Questions

For **Problems 1-10**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***one or more than one option*** may be correct. Choosing **ALL** the correct options of a problem will be treated as a **correct** answer, any other answer will be treated as a **wrong** answer. For getting credit, you have to write **ALL** the correct options, i.e. for a problem, if options **(b), (c), (d)** are correct, you mention the concerned problem number and ONLY write **(b), (c), (d)**.

For **Problems 1-10**, **1 point** will be awarded for correctly answering a problem. However, nothing will be awarded or deducted for wrong answers/unattempted problems.

Problem 1. *Let F be a field, $F[X]$ be the ring of polynomials in one variable over F. We define a F- vector space map $\delta : F[X] \to F[X]$ in the following way, $\delta(k) = 0$ for all constant polynomials k, $\delta(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n-1} (i+1)a_{i+1}x^i$ for $i \geq 1$. Let for any $f \in F[X]$, we denote $\Delta(f)$ to be the smallest integer n such that $\delta^n(f)$ ($\delta^n(f)$ means δ applied to f n times) is 0 (i.e. the zero polynomial). Suppose , for all polynomials $f \in F[X]$, $(\Delta(f) - 1)$ equals to the degree the polynomial. Then, which of the following options are correct?*

(*a*) *F cannot be a finite field*

(*b*) *F can be any infinite field.*

(*c*) *F can be any field.*

(*d*) *F should be algebraically closed.*

Problem 2. *Let S be a proper subgroup of the group $(\mathbb{R}, +)$. Suppose for all differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f^{-1}(S)$ is dense inside $\mathbb{R}$, the derivative of f vanishes at some point in $\mathbb{R}$. Which of the following options are correct?*

(*a*) *S can be the trivial subgroup.*

(*b*) *S is closed inside $\mathbb{R}$.*

(*c*) *S can be a dense subset of $\mathbb{R}$.*

(*d*) *none of the above*

Problem 3. *For a group G, we denote the commutator subgroup of G by $[G, G]$ i.e. the subgroup spanned by all elements of the form $ghg^{-1}h^{-1}$ where $g, h \in G$. For further details, see this link* `https://en.wikipedia.org/wiki/Commutator_subgroup`*. Which of these groups can be possible choices for $G/[G, G]$ where G is non-abelian ?*

(a) $\mathbb{Z}$

(b) $\mathbb{Z}/2\mathbb{Z}$

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(d) $\mathbb{Z}/4\mathbb{Z}$

Problem 4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $P_n(x) = \int_0^{x^2} f_n(t)dt, \forall n \in \mathbb{N}$.*
Which of the following options are correct?

(a) *If $\{P_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function P, then $\{f_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function f.*

(b) *If $\{f_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function f, then $\{P_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function P.*

(c) *If f_1 is uniformly continuous in $\mathbb{R}$, then P_1 is uniformly continuous in $\mathbb{R}$.*

(d) *none of the above*

Problem 5. *Let G be a graph with V as the set of its vertices and E as the set of its edges. Suppose f is a function from E to the set of integers. We call function $g : V \to \mathbb{Z}$ f-**good** if for any two neighbouring vertices $v_1, v_2 \in V$, $g(v_1) + g(v_2) = f(v_1, v_2)$ where (v_1, v_2) is the edge between v_1 and v_2. Suppose for all functions $f_1 : E \to \mathbb{Z}$, there exist infinitely many f_1-**good** functions $g_1 : V \to \mathbb{Z}$, if at all there exists one. Then, G can have a cycle of length*

(a) *3*

(b) *4*

(c) *5*

(d) *6*

Problem 6. *We call an $n \times n$ real-valued matrix A to be "$\mathbb{R}$-**spanny**" if the set $\{P^{-1}AP : P$ is an invertible $n \times n$ real-valued matrix $\}$ spans the set of $n \times n$ real matrices as an $\mathbb{R}$-vector space.*

*Similarly, we call an $n \times n$ real-valued matrix B to be "$\mathbb{Z}$-**spanny**" if every $n \times n$ real matrix can be expressed as a finite $\mathbb{Z}$-linear combination of elements in the set $\{P^{-1}BP : P$ is an invertible $n \times n$ real-valued matrix $\}$ Choose the correct alternatives.*

(a) *No $n \times n$ matrix is $\mathbb{Z}$-**spanny**.*

(b) *Every $n \times n$ matrix $\mathbb{Z}$-**spanny**.*

(c) *Every $n \times n$ matrix $\mathbb{R}$-**spanny**.*

(d) *none of the above*

Problem 7. *Let G be a non-abelian group with no proper, non-trivial, normal subgroups. How many proper, non-trivial normal subgroups does $G \times G$ have?*

(a) *2*

(b) *3*

(c) *4*

(d) *depends on the group G chosen*

Problem 8. *Let $p(x)$ be a power series with $p(x) = 1 + \sum_{i=1}^{\infty} a_i x^i,\ a_i \in \mathbb{R}$. Suppose $g(x) = \sum_{i=0}^{\infty} b_i x^i,\ b_i \in \mathbb{R}$ such that $p(x)g(x) = 1$. For all positive integers n, let $g_n(x) = \sum_{i=0}^{n} b_i x^i$. Which of the following conditions imply that $g_n(x)$ has no positive real roots for all positive integers n?*

(a) $a_i < 0\ \forall$ *odd integers* $i > 1$

(b) $a_i > 0\ \forall$ *integers* $i \geq 1$

(c) $a_i < 0\ \forall$ *integers* $i \geq 1$

(d) *none of the above*

Problem 9. *Let* $f(x,y) = \sum_{i=1}^{n} a_i x^i y^{i-1}$ *where* $a_i \in \mathbb{R}$ *with* $a_n \neq 0$*. We are given that* f *satisfies the property : For all* $y_0 \in (-1,4)$

$$\sum_{i=1}^{n-1} i a_i x_0^{i-1} y_0^{i-1} = 0 \implies \text{ for all } x \in U(x_0)\ ,\ f(x,y_0) \text{ is constant}$$

where $U(x_0)$ *is a neighborhood around* x_0
Let $g_y(x) = f(x,y)$ *for all* $y \in (-1,4)$*. Then which of the following is true:*

(a) g_y *is surjective onto* $\mathbb{R}$ *for all* $y \in (2,3)$

(b) g_y *is surjective onto* $\mathbb{R}$ *for all* $y \in (1,4)$

(c) g_y *is surjective onto* $\mathbb{R}$ *for all* $y \in (-1,1)$

(d) *None of the above*

Problem 10. *For* $n \times n$ *matrices* A *and* B *with real entries we define a function* $f_{A,B} : \mathbb{R} \to \mathbb{R}$*,* $f_{AB}(x) = det(xA + B)$*. Choose the correct options.*

(a) *if there exists matrices* A, B *with real entries such that the function* $f_{AB}(x)$ *is bounded, then* $det(A) = 0$*.*

(b) *For all matrices* A, B *with real entries, the function* $f_{AB}(x)$ *vanishes at a point.*

(c) *There exists matrices* A, B *with real entries such that the function* $f_{AB}(x)$ *vanishes at a point.*

(d) *None of the above*

Each of **Problems 11-15** correspond to a statement, which is either **true** or **false**. For getting credit, you have to correctly decide whether the Statement is true or false. For a problem, if the statement is true, you mention the problem number and write **TRUE** and if the statement is false, you mention the problem number and write **FALSE**.

For **Problems 11-15**, **1 point** will be awarded for correctly answering a problem. However, nothing will be awarded or deducted for wrong answer/unattempted problems.

Problem 11. *Suppose* f *is a continuous function from* $\mathbb{R} \to \mathbb{R}$*. Suppose* $lim_{x\to\infty} e^x f(x)$ *exists. Then* f *is uniformly continuous in* $[0, \infty)$

Problem 12. *For all positive integers* n*, let* $S^n = \{x : x \in \mathbb{R}^{(n+1)} : |x| = 1\}$*. Suppose* f_n *is a continuous , surjective function from* S^n *to the closed interval* $[0,1]$*, for every positive integer* n*. Then,* $\forall n \geq 1$*,* $\forall x \in [0,1]$*,* $f_n^{-1}(x)$ *is infinite.*

Problem 13. *There are n blue balls, n red balls and n white balls in a box, $n > 1$. You are allowed to pick up balls from the box one by one, without putting them back in the box. The probability that the last ball picked up by you is blue doesn't depend on n.*

Problem 14. *It is possible to divide a heptagon into 5 convex pentagons and 6 convex hexagons such that each of its vertices belong to at least two smaller polygons (pentagon or hexagon)*

Problem 15. *Let A be a 3×3 symmetric real matrix. Suppose that trace of A is 3, and trace of A^2 is 3. It is also given that 1 is an eigenvalue of A. Then the determinant of A can be any positive integer.*

Each problem in **Problems 16-20** has an integer valued answer. For getting credit, you have to correctly write the integer, i.e. if the answer to a problem is **5**, you write the concerned problem number and write **5** beside it. Each problem is worth **1 point**. However, **NO** points will be deducted or awarded for wrong answers/unattempted problems.

Problem 16. *Find the number of injective group homomorphisms from S_3 to S_5 (S_n denotes the permutation group on a set with n elements).*

Problem 17. *We call an integer n **basy** if for all real valued 2018×2018 matrices A such that $A^2 = -nI$ (where I is the identity matrix of size 2018×2018), there exist vectors $\{v_1, v_2, \cdots, v_{1009}\}$, $v_i \in \mathbb{R}^{2018}$ $\forall i$, such that the set $\{v_1, v_2, \cdots, v_{1009}, A(v_1), A(v_2), \cdots A(v_{1009})\}$ forms a basis for $\mathbb{R}^{2018}$. Then how many **basy** integers are there in the set $\{1, 2, \cdots, 10\}$?*

Problem 18. *Let G be an abelian group such that there exists a proper subgroup H of G such that G is **NOT** isomorphic to $H \times G/H$,i.e. the direct product of the subgroups H and G/H. (You may see this link `https://en.wikipedia.org/wiki/Direct_product_of_groups` , `https://groupprops.subwiki.org/wiki/Structure_theorem_for_finitely_generated_abelian_groups`). Let $S = \{50, 51, 52, \cdots 69, 70\}$. How many numbers in S can be the order of G?*

Problem 19. *Let $A_1, A_2, \cdots, A_9$ be 9 players playing a round-robin tournmaent (each player plays every other player exactly once). Every game results in either a win or a loss. At the end of the tournament, it is seen that the total number of wins of $A_1, A_2, \cdots, A_9$ are respectively $4, 3, 6, 4, 5, 5, 1, 4, 4$. Find the number of tuples (A_p, A_q, A_r), $p, q, r \in \{1, \cdots, 9\}$ such that player A_p has beaten player A_q, player A_q has beaten player A_r, player A_r has beaten player A_p.*

Problem 20. *Best friends Tonal Cumb and King Jongu are studying mathematics together peacefully. Jongu is smart and knows the list of composite numbers by heart, but finds prime numbers confusing. On the other hand Cumb being the brighter one can understand primes, but finds composite numbers harder to grasp. In order to help each other they have agreed to play a game. Among the integers from* 1 *to* 8*(both inclusive), each shall pick a random number and shall inform the other. Suppose Jongu has picked* x *and Cumb has picked* y*. If* x *is a prime number Jongu pays* x *Rupees to Cumb, and does nothing otherwise. In contrast, if* y *is a composite number Cumb pays* y *Rupees to Jongu, and does nothing otherwise. How much money is Jongu expected to gain from this game, if they play this once?*

Subjective Questions

Problem 1. *Given any sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.*
Statement 0 *:* $\sum_{n\in\mathbb{N}} | a_n |$ *is convergent.*
Statement 1 *: For all but finitely many bijections $f : \mathbb{N} \to \mathbb{N}$,* $\sum_{n\in\mathbb{N}} a_{f(n)}$ *is convergent.*
Statement 2 *: For infinitely many bijections $f : \mathbb{N} \to \mathbb{N}$,* $\sum_{n\in\mathbb{N}} a_{f(n)}$ *is convergent.*
Prove or disprove the following :

1. ***Statement 1*** $\implies$ ***Statement 0***
2. ***Statement 2*** $\implies$ ***Statement 0***.

Problem 2. *Given a function $f : \mathbb{R}^2 \to \mathbb{R}$. Prove or disprove the following :*

1. *If $f \circ \nu$ is continuous at 0 for all polynomial curves ν (see definition below) with $\nu(0) = (0, 0)$, then f is continuous at $(0, 0)$.*
2. *If $f \circ \nu$ is continuous at 0 for all curves ν (see definition below) with $\nu(0) = (0, 0)$, then f is continuous at $(0, 0)$.*

   Definition 1. *A curve in $\mathbb{R}^2$ is a continuous function $\mu : \mathbb{R} \to \mathbb{R}^2$.*
   Definition 2. *A curve μ is said to be a polynomial curve if $\mu(x) = (x, P(x))\ \forall x \in \mathbb{R}$ for some polynomial $P(x) \in \mathbb{R}[x]$.*

Problem 3. *Let $n = p + 1$ where p is an odd prime and $S = \{(i, j)|1 \leq i < j \leq n\}$. Consider a graph with n vertices labelled as $1, 2, \cdots, n$. Let t_n be the number of subsets $T \subset S$ such that the graph formed by connecting the vertices with edges given by the pairs in T is connected. Prove that $t_n^2 \equiv 1 \pmod{p}$.*

Problem 4. *Let G be a group, $n \in \mathbb{N}$. A tuple (G, n) is said to be good if there exists a homomorphism $\phi : G \to S_n$ such that $\phi(G)$ does not fix any $i \in \{1, \cdots, n\}$, that is, $\forall$ $i \in \{1, \cdots, n\}$, $\exists\ g \in \phi(G)$ such that $g(i) \neq i$.*
*Let G' be a finite abelian group. Let (G', n) be good for all but finitely many $n \in \mathbb{N}$. Find all possible values of $|G'|$. (You may refer to the links given in **Problem 18**.)*

Problem 5. *Let p be a given prime number. Define $f(p, n)$ to be the least positive integer* *so that* $\exists\ P(x) \in \mathbb{Z}[X]$*, with* $P(X)$ *being monic and* $\deg(P) = f(p, n)$ *such that* $p^n \mid P(m)$ $\forall\ m \in \mathbb{Z}$*. Prove that*

$$\lim_{n\to\infty} \frac{f(p,n)}{n} = p - 1$$

Problem 6. *Let* G *be a connected graph. A subset of its edges is called bunny if each vertex has even degree in the subgraph generated by the elements of that subset. Prove that the number of bunny subsets is a function of the number of edges and the number of vertices of* G*.*
(To be precise, prove that if G_1 *and* G_2 *are two connected graphs with same number of edges and vertices, then they have the same number of bunny subsets.)*