TESSELLATE PRESENTS

Scholastic Test of Excellence in Mathematical Sciences

(S.T.E.M.S.)

Title Sponsor

BRILLIANT
Math and science done right

Mathematics Exam

School Category

January 14, 2018

Message from The S.T.E.M.S. Team

Dear applicant,

Greetings from team S.T.E.M.S. This is the first edition of Scholastic Test of Excellence in Mathematical Sciences. All that we have to say before the exam is: DO NOT BE NERVOUS! The questions can be too hard for some of you, and too easy for the rest. Remember, the aim should not be to solve every question completely. Rather, try to attempt the questions, and write whatever you are thinking, even if it is a couple of lines. If you think that you are not being able to solve the questions, and they are too hard, well then they will be hard for everyone and the ranking will be relative. We are NOT looking for exactly correct answers for all the questions! Rather, we would like to see what you are thinking. We shall be as lenient as possible with our grading, and try to give as much part marks as possible in every subjective question.

Best wishes! Rock the exam.

BRILLIANT
Math and science done right

Rules and Regulations

Submission of Answer sheets

1. The digital copy of the answer sheet has to be mailed to `stems.mathsch.a1@gmail.com`. The submissions should be sent strictly within the time limit.
2. You are allowed to scan/take pictures of your answer sheet for submission. You may use TeX if you wish to submit a soft copy of the solution as a pdf file.
3. Ensure that your submitted copy is clear and readable. Illegible submissions such as bad handwriting and unclear photographs will be disqualified.
4. Send the entire answer sheet in one single mail. Repeated mails of solutions will not be accepted.
5. Email back the answer sheet from the same email id you registered from (this is the same id that you received the question paper in).
6. Mention your class in your answer sheet. Do NOT write your name or school/location in your answer sheet. Write your registration ID if you have it.

Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question is worth **1 point** and there is no negative marking.
3. You are not required to show your work for the objective part of the paper.
4. Each subjective problem is worth **20 points**.
5. For getting full credit in the subjective questions you need to give the detailed solutions. However, credit will also be awarded for partially correct solutions.
6. There is no negative marking in the subjective section as well.
7. **The subjective part will be graded only if you score above a certain cut-off (which will be decide later) in the objective section of the paper. However, for the final score, your total score (subjective + objective) will be taken into consideration.**

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.
4. You are not allowed to post/discuss the problems in any online forum within the exam time.
5. For any querries regarding the paper, please email `stems.school.math@gmail.com`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `stems.tessellate@gmail.com` .

Notation

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ are the set of integers, rationals, positive integers, real numbers and positive real numbers respectively.
2. $\in$ denotes "**belongs to**" , $\forall$ denotes "**for all**", $\exists$ denotes "**there exists**", $\nexists$ denotes "**there does not exist**"
3. $\sum\limits_{\text{cyc}}$ denotes cyclic sum, the definition of cyclic sums is given at `https://artofproblemsolving.com/wiki/index.php?title=Cyclic_sum`
4. $a \mid b$ for any two integers a, b with $a \neq 0$ means that a is a factor of b e.g. 2 is a factor of 6.
5. $a \nmid b$ means that a is **NOT** a factor of b. Example : $2 \nmid 5$ is true
6. "**iff**" denotes "**if and only if**" . For example, "Statement 1 is true" **iff** "Statement 2 is true" means that "Statement 1 is true" implies "Statement 2 is true" and vice-versa.
7. $\sum, \prod$ denote summation and product respectively, for further clarification, visit `https://math.illinoisstate.edu/day/courses/old/305/contentsummationnotation.html`

Objective Questions

For **Problems 1-10**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***one or more than one option*** may be correct. Choosing **ALL** the correct options of a problem will be treated as a **correct** answer, any other answer will be treated as a **wrong** answer. For getting credit, you have to write **ALL** the correct options, i.e. for a problem, if options **(b), (c), (d)** are correct, you mention the concerned problem and ONLY write **(b), (c), (d)**.

For **Problems 11-16**, write **ONLY** the numerical value asked alongside the question number, **NO** justification for the answers is required for the problems in this section.

For **Problems 1-16**, **1 point** will be awarded for correctly answering a problem, **NO** negative marks shall be awarded for wrong answers/unattempted problems .

Problem 1. *We call an integer m* ***split*** *if the set $S(m) = \{1, 2, 3, \cdots 2m\}$ can be split into m disjoint sets each containing two elements such that the sum of the two numbers in each set is a prime. Which of the following options are correct?*

(a) *There are infinitely many* ***split*** *integers of the form $4k + 1,\ k \in \mathbb{Z}$.*

(b) *There are only finitely many* ***split*** *integers of the form $4k,\ k \in \mathbb{Z}$.*

(c) *All integers of the form $4k + 3,\ k \in \mathbb{Z}$ are* ***split****.*

(d) *None of the above*

Problem 2. *Let $f(n) = n^2 - n - 3$. Choose the correct options.*

(a) *There are infinitely many integers n such that $13 \mid f(n)$.*

(b) *There are infinitely many integers n such that $13 \nmid f(n)$.*

(c) *There are infinitely many integers n such that $13^2 \nmid f(n)$.*

(d) *There are only finitely many integers n such that $13^2 \nmid f(n)$.*

Problem 3. *Alice and Bob are playing a game. At the beginning of the game, there is a cubic polynomial with integral coefficients written on the blackboard, which we denote as the* ***starting polynomial****. The two players take turns one by one. In each turn, the player* ***either*** *chooses any natural number n and replaces the existing cubic polynomial $f(x)$ on* *the blackboard by any one of $f(n+x)$, $f(nx)$, $f(x)+n$* ***or*** *just changes the sign of the coefficients of x^2, i.e. if the existing polynomial is $a_0+a_1x+a_2x^2+a_3x^3$, then he can replace it by $a_0+a_1x-a_2x^2+a_3x^3$. Alice takes the first turn. Bob wins if, after finitely many moves, the cubic polynomial on the blackboard has all coefficients (upto x^3) non-zero and equal. For which of the* ***starting polynomials*** *can Alice ensure that Bob does not win in finite number of moves?*

(a) x^3+4x^2+4x+1

(b) x^3+6x^2+7x

(c) x^3+3

(d) *none of the above*

Problem 4. *Let $BACD$ be a convex cyclic quadrilateral in which AD bisects $\angle BAC = 60°$, $|AB| = 1$, $|AC| = 2$. Then $|AD|$ is:*

(a) $\sqrt{3}$

(b) $\sqrt{2}$

(c) *1*

(d) $\sqrt{5}$

Problem 5.

Let J be a point in the plane of a given $\triangle ABC$ with $|AB| = 2018$, $|AC| = 2017$, $|BC| = 2016$ such that pedal triangle of J with respect to $\triangle ABC$ is equilateral and is also having the minimal area out of all the equilateral triangles that have their vertices on $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ (one on each of the sides). Then $\dfrac{|JB|}{|JC|}$ is:

(a) $\dfrac{2018^2}{2017^2}$

(b) $\dfrac{2018}{2017}$

(c) $\dfrac{2017}{2018}$

(d) $\dfrac{2016}{2017}$

Problem 6. *$A \subseteq \mathbb{Q}$ is such that $\forall\ z \in A$, we have:*

- $(z+1) \in A$

- $(z-1) \in A$
- $\frac{-1}{z} \in A$ *(if $z \neq 0$)*
- $1 \in A$

Let $S = \{t \mid \frac{p}{q} \in A, gcd(p,q) = 1, t = p^2 + q^2\}$. *Find the number of elements of S that are atmost 100. (that is, find* $\mid S_{100} \mid$ *where* $S_{100} = \{x \mid x \in S, x \leq 100\}$*)*

(a) *23*

(b) *22*

(c) *24*

(d) *21*

Problem 7. *Find the total number of possible injective functions* $f : \mathbb{R} \to \mathbb{R}$ *such that:* $f(3x)^4 + 2 \leq f(x^2+2)^2 + 2f(3x)$.

(a) *0*

(b) *1*

(c) *2*

(d) ∞

Problem 8. *Let f be a formal power series with every coefficient either 0 or 1; with* $f\left(\frac{2}{3}\right) = \frac{2017}{2^{2018}}$. *What is the period of decimal expansion of* $f\left(\frac{1}{2}\right)$.

(a) *2017*

(b) *2018*

(c) 2^{2018}

(d) ∞ *(that is,* $f\left(\frac{1}{2}\right)$ *is irrational)*

Problem 9. *Find the number of polynomials with degree at − most 2018 (less than or equal to 2018) such that* $P(x^2) = P(x) \cdot (P(x) + 2x)\ \forall\ x$.

(a) *2019*

(b) *2018*

(c) *1009*

(d) ∞

Problem 10. *Find the number of pairs* $\{a, b\}$ *such that* $a+b = 2017$*;* $a, b \in \mathbb{N}$ *and there is no carry over while adding* a *and* b *in base 10.*

(a) *48*

(b) *47*

(c) *49*

(d) *50*

Problem 11. *Given* $\triangle ABC$*,* P *be foot of* A*-angle bisector and* OH*,*BC *intersect at* X*. If* $AX \perp PH$*, then find all possible values of* $\angle A$
Note. O *is the circumcentre of* $\triangle ABC$*,* H *is orthocentre of* $\triangle ABC$*.*

Problem 12. $a, b, c, d \in \mathbb{N}$ *(not all equal). Find the least possible value of* $a^4 + b^4 + c^4 + d^4 - 4abcd$*.*

Problem 13. $A = \{1, 2, \cdots, n\}$*,* $n \geq 2$*. Find a closed form (that is,* ***free*** *of any summation* $\sum$ *or product symbols* $\prod$ *in its expression) of the number of ways of choosing* $P_1, P_2, \cdots, P_k \subseteq A$ *such that* $| P_1 \cap P_2 \cdots \cap P_k | \geq 2$*.*

Problem 14. *Find the least possible value of*

$$\frac{\prod_{i=1}^{5}(a_i^7 + a_i^3 - a_i^2 - 3a_i + 7)^{\frac{1}{5}}}{\sum_{i=1}^{5} a_i}$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^+$*.*

Problem 15. *Find all possible 6-digit numbers* n *(in base 10) such that* $n, 2n, 3n, 4n, 5n, 6n$ *have the same set of digits and the digits appear in the same cyclic order for all of them. (If you think there are no such numbers, answer- '*ϕ*')*

Problem 16. *Given a square* $ABCD$*.*
Find the number of possible positions of point P *(both* ***inside*** *and* ***outside*** *of the given square) on the perpendicular bisector of* AB *such that the angles* $\angle PAB$*,* $\angle PBA$*,* $\angle PCD$*,* $\angle PDC$ *are in the set* $S = \{\frac{\pi}{m} \mid m \in \mathbb{N}\}$*.*

Subjective Problems

Problem 1. *Given a prime* $p \equiv 3 \pmod 4$. *Prove that for all given integers* a, b *with* $p \nmid gcd(a,b)$, $\exists\ c, d \in \mathbb{Z}$ *so that* $p \mid ac - bd - 1$ *and* $p \mid ad + bc$.

Problem 2. $a_1, \cdots, a_{2018} \in \mathbb{R}^+$ *such that* $\prod_{i=1}^{2018} a_i = 1$. *Prove that :*

$$\sum_{cyc} \frac{(a_1 \cdots a_{2017})^{2016}}{a_1^{2017^2+1} + \cdots + a_{2017}^{2017^2+1} + (a_1 \cdots a_{2017})^{2017^2+1}} \leq 1$$

Problem 3. *Given points* A, B *with* $\mid AB \mid = 7m$, *a point* P *is called good if the length of* $A-$*median and* $B-$*altitude are equal in* $\triangle PAB$. *Find the maximum possible distance between any two good points.*

Problem 4. S *is a finite subset of* $\mathbb{R}^2$. *Such that :*

- *No three points of* S *are collinear.*
- *If* $A, B, C \in S$; *then* $A', B', C' \in S$ *where* A', B', C' *are diametrically opposite to* A, B, C *(respectively) with respect to the circumcircle of triangle* ABC.

Prove that all points in S *are concyclic.*

Problem 5. *Prove that given any polynomial* $P(X)$ *with integral coefficients,* $\nexists M \in \mathbb{N}$ *such that* $P(n)$ *is square-free* $\forall\ n \geq M$.

Definition. $T \in \mathbb{N}$ *is said to be square-free iff* $p^2 \nmid T\ \forall$ *primes* p.

Problem 6. *Let* $n = p + 1$ *where* p *is an odd prime. A group of* n *scientists attend a seminar. Some of them shake hands with each other. Two scientists* A, B *are said to be connected if either of the following are true:*

1. A *and* B *shake hands.*
2. *There are scientists* $S_1, \cdots, S_k$ *for some* $k \geq 1$ *such that* A *shakes hands with* S_1, S_i *shakes hands with* $S_{i+1}\ \forall\ 1 \leq i \leq (k-1)$ *and* S_k *shakes hands with* B.

Let t_n *be the number of such scenarios in which scientists have shook hands in such a way that any two scientists attending the seminar are connected. Prove that* $t_n^2 \equiv 1 \pmod p$.

Definition. *Two scenarios are said to be different if there exists a pair of scientists who shook hands in one scenario but not in the other.*