TESSELLATE PRESENTS

Scholastic Test of Excellence in Mathematical Sciences (S.T.E.M.S.)

TITLE SPONSOR

BRILLIANT

Math and science done right

Physics Exam

College Category

January 13, 2018

Message from The S.T.E.M.S. Team

Dear applicant,

Greetings from team Tessellate. This is the first edition of Scholastic Test of Excellence in Mathematical Sciences. All that we have to say before the exam is: DO NOT BE NERVOUS! The questions can be too hard for some of you, and too easy for the rest. Remember, the aim should not be to solve every question completely. Rather, try to attempt the questions, and write whatever you are thinking, even if it is a couple of lines. If you think that you are not being able to solve the questions, and they are too hard, well then they will be hard for everyone and the ranking will be relative. We are NOT looking for exactly correct answers for all the questions! Rather, we would like to see what you are thinking. We shall be as lenient as possible with our grading, and try to give as much part marks as possible in every subjective question.

Best wishes! Rock the exam.

Rules and Regulations

Submission of Answer sheets

1. The digital copy of the answer sheet has to be mailed to `stems.phycol@gmail.com`. The submissions should be sent strictly within the time limit.
2. You are allowed to scan/take pictures of your answer sheet for submission. You may use TeX if you wish to submit a soft copy of the solution as a pdf file.
3. Ensure that your submitted copy is clear and readable. Illegible submissions such as bad handwriting and unclear photographs will be disqualified.
4. Send the entire answer sheet in one single mail. Repeated mails of solutions will not be accepted.
5. Email back the answer sheet from the same email id you registered from (this is the same id that you received the question paper in).
6. Do NOT write your name or school/location in your answer sheet. Write your registration ID if you have it.

Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question has **4 options** out of which only **one is correct**, and is worth **5 marks**.
3. You are not required to show your work for the objective part of the paper.
4. For the objective section, write your answers in the form of a tabular column where the rows denote the question number in order and the columns options *(a)* to *(d)* and tick the appropriate grids.For instance if you want to mark the answer (a) for question 1, and option (b) for question 10, then this is how the corresponding row would look like:

	(a)	(b)	(c)	(d)
(1)	✓			
		⋮		
(10)		✓		

5. Each **subjective problem** is worth **10 marks** each.
6. **The subjective problems will be graded only if you score above a certain cut-off (which will be decided later) in the objective section of the paper.**
7. For the subjective questions you need to formally show your reasoning. Marks will be awarded for partially correct solutions.
8. There is no negative marking.

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.
4. You are not allowed to post/discuss the problems in any online forum within the exam time.
5. For any querries regarding the paper, please email `stems.college.physics@gmail.com`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `stems.tessellate@gmail.com` .

Objective Questions

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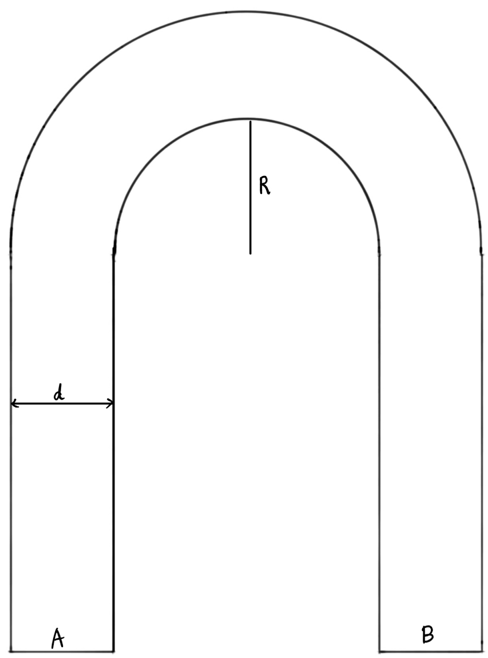
Problem 1. *Suppose operators* $\hat{\mathbf{A}}$ *and* $\hat{\mathbf{B}}$ *are given by* $\hat{\mathbf{A}} = \frac{x^2}{2}$ *and* $\hat{\mathbf{B}} = -\frac{i\hbar}{x}\frac{\partial}{\partial x}$. *Find the least value* $\sigma_A\sigma_B$ *can take (where* σ_A *and* σ_B *are the standard deviations of* $\hat{\mathbf{A}}$ *and* $\hat{\mathbf{B}}$ *respectively).*

$(a)\dfrac{\hbar^2}{4}$ $(b)\dfrac{\hbar^2}{3}$ $(c)\dfrac{\hbar^2}{2}$ $(d)\hbar^2$

Problem 2. *Consider a particle of mass* m *near the surface of the earth falling under the influence of earth's gravitational field. The particle is at* (x_0, y_0, z_0) *at time* $t = 0$ *and at* $(x_0, y_0, z_0 - \frac{g}{2})$ *at time* $t = 1$. *Find the action* $\mathcal{S}_{cl}$ *of the particle corresponding to the classical motion between the points* (x_0, y_0, z_0) *and* $(x_0, y_0, z_0 - \frac{g}{2})$.

$(a) mg^2 - mgz_0$ $(b) m\sqrt{gz_0^3}$ $(c)\dfrac{m\sqrt{gz_0^3}}{3}$ $(d)\dfrac{mg^2}{3} - mgz_0$

Problem 3. *A glass rod of rectangular cross-section is bent into the shape of a horse-shoe as shown in the figure below. Suppose a beam of light is falling perpendicular on the surface A. Then what is the minimum value of* $\frac{R}{d}$ *for which the whole beam of light emerges through the surface B, given the refractive index of glass to be* 1.5.



$(a)1.5$ $(b)2$ $(c)2.5$ $(d)3$

Problem 4. *Consider a particle of mass m under a central potential* $V = -\frac{k}{r}$. *Where* k *is some positive constant and* $r = \sqrt{x^2 + y^2 + z^2}$. *Suppose the energy* E *of the particle is negative. Consider the vector* $\mathbf{A} = \frac{1}{\sqrt{-2mE}}(\mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}})$. *Find the Poisson bracket relations* $\{A_i, A_j\}$.

$(a)0$ $(b)\sum_{k=1}^{3}\epsilon_{ijk}A_k$ $(c)\sum_{k=1}^{3}\epsilon_{ijk}L_k$ $(d)\delta_{ij}$

Problem 5. *Gauss' law would be invalid if*

(a) there were magnetic monopoles

(b) the inverse-square law were not exactly true

(c) the velocity of light were not a universal constant

(d) None of the above

Problem 6. *Suppose an atom could exist in two states, a ground state of mass M and an excited state of mass $M + m$. If the atom gets excited by absorbing a photon, what should be the photon frequency in the frame in which the atom is at rest initially.*

$$(a)\frac{mc^2}{h}\left(1+\frac{m}{2M}\right) \qquad (b)\frac{Mc^2}{h}\left(1+\frac{m}{2M}\right) \qquad (c)\frac{mc^2}{h}\left(1+\frac{M}{2m}\right) \qquad (d)\frac{Mc^2}{h}\left(\frac{m}{M}+\frac{1}{2}\right)$$

Problem 7. *How does the electron density of an atom change as the distance from the atom increases?*

(a)Decreases like a Gaussian

(b) Increases linearly till a certain distance and then decreases exponentially

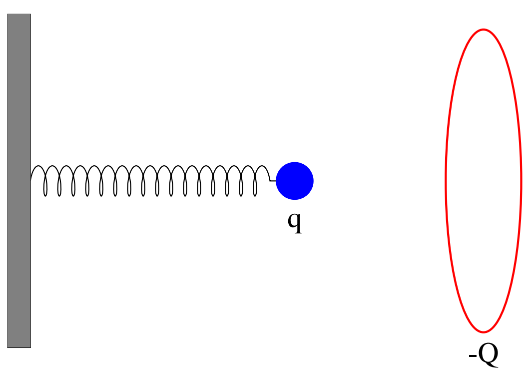
(c) Decreases exponentially

(d) Oscillates with slowly decreasing amplitude

Problem 8. *Suppose there is a system with three energy levels 0, ϵ and 8ϵ at temperature T. Calculate the average energy in the limit $T \longrightarrow 0$.*

$$(a)0 \qquad (b)3\epsilon \qquad (c)2.5\epsilon \qquad (d)0.5\epsilon$$

Problem 9. *Consider a particle of mass m and charge q attached to a spring with spring constant k and constrained to move along x-axis. Suppose at equilibrium the particle is at origin. Now a uniformly charged ring of radius R with charge $-Q$ and x-axis as its axis is placed such that the center of the ring is located at $(L, 0, 0)$. (Assume $q, Q > 0$) Find the new equilibrium point of the particle.*



$(a) L - \sqrt{\left(\frac{qQ}{4\pi\epsilon_0 k}\right)^{2/3} - R^2}$ $(b) L - \sqrt{\left(\frac{qQ}{4\pi\epsilon_0 k}\right)^{2/3} + R^2}$

$(c) L + \frac{1}{2}\sqrt{\left(\frac{qQ}{4\pi\epsilon_0 k}\right)^{2/3} - R^2}$ $(d) L - \frac{1}{2}\sqrt{\left(\frac{qQ}{4\pi\epsilon_0 k}\right)^{2/3} + R^2}$

Problem 10. *A cubical block of side a and density ρ is immersed in a tank filled with two liquids , A and B with densities $\frac{\rho}{2}$ and 2ρ respectively. The height of liquid A is $2a$ and the height of liquid B is $4a$. The system comes to equilibrium after some time. Once the system is in equilibrium the block is slowly pulled out of the tank. Calculate the amount of work done in pulling the block out of the tank.*

$(a) \frac{5\rho g a^4}{4}$ $(b) \frac{7\rho g a^4}{4}$ $(c) \frac{3\rho g a^4}{2}$ $(d) \frac{5\rho g a^4}{2}$

Problem 11. *For a quantum harmonic oscillator of mass m and frequency ω introduce a term λx in the Lagrangian and find the partition function where $\beta = \frac{1}{k_B T}$.*

$(a) Z = \frac{e^{-\beta\hbar\omega/2 - \frac{\beta\lambda^2}{2m\omega}}}{1 - e^{-\beta\hbar\omega/2}}$ $(b) Z = \frac{e^{-\beta\hbar\omega/2 - \frac{\beta\lambda^2}{m\omega}}}{1 - e^{-\beta\hbar\omega/2}}$

$(c) Z = \frac{e^{-\beta\hbar\omega/2 - \frac{\beta\lambda^2}{4m\omega}}}{1 - e^{-\beta\hbar\omega/2}}$ $(d) Z = \frac{e^{-\beta\hbar\omega/2 - \frac{\beta\lambda}{2m\omega^2}}}{1 - e^{-\beta\hbar\omega/2}}$

Problem 12. *For a particle in a ring, $\psi_n(\theta) = \frac{e^{\pm in\theta}}{\sqrt{2\pi R}}$ with n a nonnegative integer, find the expectation value of the x coordinate of the particle on the ring in the .*

$(a) 0$ $(b) R/2$ $(c) 2\pi$ (d) *Undefined*

Problem 13. *Consider the 4-D vector space where each point represents (P, V, N, T), where P, V, N, and T are the pressure, volume, number of molecules and temperature of a system. Take 2 points A and B on the equilibrium manifold (the 3-D surface that satisfies the equation of state) and consider the following statements :*

(i)Each thermodynamic process with A and B being the initial and final states respectively can be represented by a curve connecting points A and B

(ii) Each reversible process with A and B being the initial and final states respectively can be represented by a curve that lies on the equilibrium manifold connecting points A and B

(iii) Irreversible processes from A to B need not be represented by any curve connecting A and B

(iv) Given a path P on the equilibrium manifold conneting A and B ,$S(B) - S(A) = \int_P \frac{\delta Q}{T}$, where S(K) is the entropy of the system at the point K.

(v) Given any path P connecting A and B, $S(B) - S(A) = \int_P \frac{\delta Q}{T}$

(vi)$S(B) - S(A) = \int_{rev} \frac{\delta Q}{T}$, and is a constant no matter what the physical process is

Which of the above statements are false?

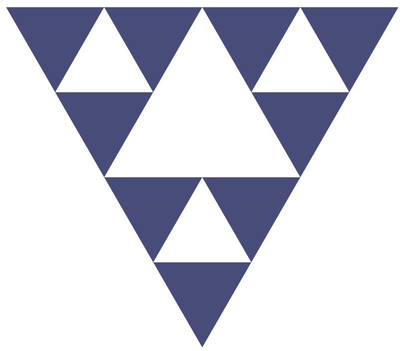
$(a) (ii), (v), (vi)$ $(b) (i), (ii), (v)$ $(c) (i), (v)$ $(d) (iii), (iv), (vi)$

Problem 14. *For the Hamiltonian $H = \lambda S_z B_z$, where B_z and λ are constants, assuming that the system starts in the S_x+ eigenstate, what is the probability of finding it in the S_x- eigenstate at time t?*

$(a) sin^2(B_z \lambda t)$ $\quad$ $(b) cos^2(B_z \lambda t)$ $\quad$ $(c) 1$ $\quad$ $(d) \frac{t}{2}$

Subjective Questions

Problem 15. *Take an equilateral triangular region of side a with uniform mass density μ, and remove the middle equilateral triangle of side $\frac{a}{2}$, now remove the middle equilateral triangle from the remaining three triangles, and so on. Find the moment of inertia of this object, around an axis through its center and perpendicular to its plane.*



Problem 16. *We are familiar with the famous Stern-Gerlach experiments on spin-$\frac{1}{2}$ particles. But the experiments doesn't seem so amusing if you are doing it after the discovery of the fact that there are two different spins for spin-$\frac{1}{2}$ particles and it is quite intuitive that measuring the spin of the particle along (say) the $\hat{z}$ direction while it is in the $\frac{\hbar}{2}$ eigenstate of the S_x operator (spin along the $\hat{x}$ direction) gives us a 50% of chance finding it to be in the spin up or the spin down state.*

Now suppose you a have a huge collection of (massive) spin-2 particles, all in the $2\hbar$ eigenstate of the S_x operator and you send all these particles through a Stern-Gerlach apparatus whose magnetic field is oriented along the $\hat{z}$ direction. Find the probability of particles that will be in the $2\hbar, \hbar, 0, -\hbar, -2\hbar$ eigenstates of the S_z operator after passing through the apparatus.

Problem 17. *A pendulum consists of a mass m at the end of a massless stick of length l. The other end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos(\omega t)$, where $A << l$. It turns out that if ω is large enough, and if the pendulum is initially nearly upside-down, then it will, surprisingly, not fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position. Explain why the pendulum doesn't fall over, and find the frequency of the back and forth motion.*

Problem 18. *Consider the diagram given below. A cuboidal box is divide by a heavy movable piston with the initial conditions as given above in the diagram. Both A and B are filled with the same number of molecules of a single type of gas (not necessarily ideal). As the system is not in physical equilibrium (initially), it leads to the movement of the piston, which is assumed to move slow enough that the sub-systems of A and B(individually) along the process are in equilibrium. A and B are assumed to be in thermal equilibrium with respect to each other.(Assuming instant conduction)*

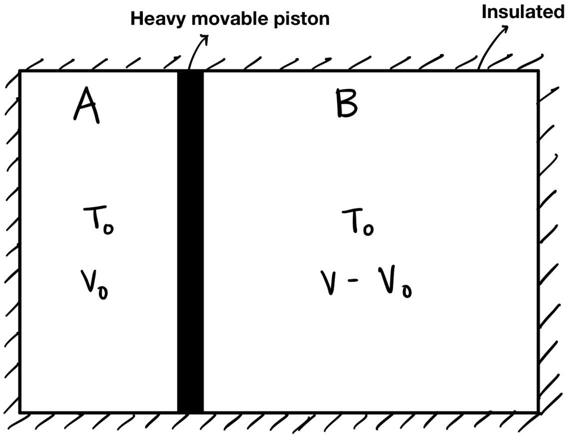


Fig: Cuboidal box with initial conditions

(i) Prove that the entropy of the system extremises when A and B are in total equilibrium with respect to each other.
(ii)Prove the relation

$$\frac{dT}{dV_A} = \frac{-T\left(\frac{\partial P(V_A,T)}{\partial T} - \frac{\partial P(V-V_A,T)}{\partial T}\right)}{\left(\frac{\partial U(V_A,T)}{\partial T} + \frac{\partial U(V-V_A,T)}{\partial T}\right)}$$

where V_A is the instantaneous volume of A,V is the total volume of the system,U is the internal energy and P is the pressure.

Problem 19. *If $\vec{E}$ and $\vec{B}$ are normal to each other and $|\vec{E}| < |\vec{B}|$, find the velocity $\vec{v}$ of a system within which $\vec{E} = 0$*

Problem 20. *(i) A uniformly charged non-conducting cylinder of radius of a with charge per unit length λ. It is wound with current carrying wire with current I with N turns per centimeter along the cylinder, such that $\frac{dI}{dt} = Kt$. Find the torque per unit length acting on the cylinder.*

(ii) Now, suppose $I = \eta t^4$, find $E_\phi(s)$ [where $E_\phi(s)$ is the electric field along the azimuthal direction, supposing the cylinder is along the z-axis and s is the distance from the z-axis such $s < a$]. Assume

$$\left.\frac{\partial^k B_z}{\partial t^k}\right|_a = b_k$$

(iii) Assuming current to be any arbitrary function of time. Write down the recurrence relation for the k^{th} derivative of E_ϕ which can be used to calculate E_ϕ in terms of b_n and $\frac{d^n I}{dt^n}$ where $n \in N$. (No need to calculate E_ϕ itself.)