TESSELLATE PRESENTS

Scholastic Test of Excellence in Mathematical Sciences

(S.T.E.M.S.)

Title Sponsor

BRILLIANT

Math and science done right

Computer Science Sample Paper

School Category

2019

Message from The S.T.E.M.S. Team

Dear applicant,

Greetings from team Tessellate. This is the first edition of Scholastic Test of Excellence in Mathematical Sciences. All that we have to say before the exam is: DO NOT BE NERVOUS! The questions can be too hard for some of you, and too easy for the rest. Remember, the aim should not be to solve every question completely. Rather, try to attempt the questions, and write whatever you are thinking, even if it is a couple of lines. If you think that you are not being able to solve the questions, and they are too hard, well then they will be hard for everyone and the ranking will be relative. We are NOT looking for exactly correct answers for all the questions! Rather, we would like to see what you are thinking. We shall be as lenient as possible with our grading, and try to give as much part marks as possible in every subjective question.

Best wishes! Rock the exam.

Rules and Regulations

Submission of Answer sheets

1. The digital copy of the answer sheet has to be mailed to `xxxxx@yyyy.zzz`. The submissions should be sent strictly within the time limit.
2. You are allowed to scan/take pictures of your answer sheet for submission. You may use TeX if you wish to submit a soft copy of the solution as a pdf file.
3. Ensure that your submitted copy is clear and readable. Illegible submissions such as bad handwriting and unclear photographs will be disqualified.
4. Send the entire answer sheet in one single mail. Repeated mails of solutions will not be accepted.
5. Email back the answer sheet from the same email id you registered from (this is the same id that you received the question paper in).
6. Do NOT write your name or college/location in your answer sheet. Write your registration ID if you have it.

Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question is worth 10 marks each.
   - If you get mark exactly the set of correct answers as correct, you get 10 marks
   - Otherwise, you get 2 points for every problem which you marked correct and are actually correct answers and loose one point for every problem which you marked
3. You are not required to show your work for the objective part of the paper.
4. For Objective questions having choice, each problem has at least one correct answer: *however, there might be many.* Choose ALL that apply, unless indicated otherwise.
5. For Problems 1-8, write your answers in the form of a tabular column where the rows denote the question number in order and the columns options *(a)* to *(e)* and tick the appropriate grids. For questions with no option (e), ignore the column with option e. For instance if you want to mark the answers $(a), (d)$ for question 1, and option

$(c),(d)$ for question 7, then this is how the corresponding row would look like:

	(a)	(b)	(c)	(d)
(1)	✓			✓
			⋮	
(7)			✓	✓

6. Each subjective problem is worth 20 marks each.
7. For the subjective questions you need to formally show your reasoning. Marks will be awarded for partially correct solutions.
8. The subjective problems will be graded only if you score above a certain cut-off (which will be decided later) in the objective section of the paper.
9. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. You are not allowed to post/discuss the problems in any online forum within the exam time.
4. For any querries regarding the paper, please email `xxxxx@yyyy.zzz`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `xxxxx@yyyy.zzz`.

Best Wishes for your Exam!

Objective Questions

Problem 1. *You are given the following list : [5,2,15,6,7,5,13] . You want to sort it in non-decreasing order. The only operation you are allowed to make is to move one element of the list to any other position in the list, that is, before the first element, after the last element, or between any two consecutive elements. The cost of a single operation is the value of the element displaced. What is the minimum total cost required to sort the list?*

(a) 18

(b) 20

(c) 22

(d) 25

Problem 2. *There are 14 people seated in a round table in a restaurant . Two adjacent persons can share meals. Each person has a certain cost for the meal he/she ordered. What is the minimum cost by which all the people can have food, (that is for every pair of adjacent persons, at least one must be served food) if the prices for the 14 people are* $13, 4, 2, 3, 1, 5, 4, 18, 17, 19, 5, 6, 9, 1$ *(Remember that the first and the last persons are adjacent).*

(a) 39

(b) 40

(c) 49

(d) 46

Problem 3. *Let* $f, g, h \colon \mathbb{N} \to \{0, 1\}$ *be functions defined as follows:*

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ g(n-1) & \text{otherwise} \end{cases} \qquad g(n) = \begin{cases} 1 & \text{if } n = 0 \\ f(n-1) & \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 - h(n-1) & \text{otherwise} \end{cases}$$

For how many values of $n \in \mathbb{N}$ *do we have* $h(n) \neq f(n)$*?*

(a) 0

(b) 1

(c) 2

(d) 3

Problem 4.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

where f_n is the n^{th} Fibonacci number.

(a) *This is true for all natural number n.*

(b) *This is true for all even number n.*

(c) *This is true for all odd number n.*

(d) *This is only true for all number $n \leq 6$.*

Problem 5. *Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be functions defined as follows:*

$$f(n,k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & k{>}n \\ f(n-1,k-1) + f(n-1,k) & \text{otherwise} \end{cases} \quad g(n) = \begin{cases} 1 & \text{if } k = 0 \\ g(n,k-1) * n & \text{otherwise} \end{cases}$$

$\sum_{k=0}^{n} f(n,k)g(2,k)$ *is equal to which of thw following:*

(a) $n^3 + 1$

(b) 3^n

(c) $2^{(n+1)} + 1$

(d) $3^{\lfloor \frac{n^2+1}{2} \rfloor}$

Problem 6. *You are a great footballer, and will be representing India in FIFA World Cup 2026. :)*
Suppose India play France in the Final, and the final scoreline reads "India 51 – France 49". What is the probability that the goal tally of India is strictly greater than the goal tally of France throughout the game.

(a) 1/2

(b) 1/49

(c) 1/50

(d) 1/100

Problem 7. *Let F_n be the number of subsets A_i of $A = [n]$, such that A_i contains no two consecutive elements of A.*
And let f_n be the n^{th} Fibonacci number then which of the following is true:

(a) 2^{n-1}

(b) f_{n+2}

(c) 3^{n-1}

(d) $\binom{m+\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}$

Problem 8. *A binary relation R, defined on a set $A = \{1, 2, \ldots, n\}$ is a subset of $A \times A$. A relation R is called anti-symmetric if aRb and $bRa \Rightarrow a = b$. Find the probability of a relation, to be anti-symmetric, chosen at random, out of total binary relations defined on A.*

(a) $3^{\frac{n(n+1)}{2}}$

(b) $3^{\frac{n(n-1)}{2}}$

(c) $3^{\frac{n(n+1)(2n+1}{2}}$

(d) $3^{\frac{[n(n+1)]^2}{2}}$

Problem 9. *You are given a graph with infinite number of vertices. The vertices are number* $1, 2, 3, \ldots$ *. There exist two edges from* i *to* $2i$ *and* $2i + 1$*. Vertex numbered 1 has weight 1. If vertex number* i *has weight* w*, then* $2i$ *and* $2i + 1$ *has weight* $\frac{w}{2}$ *each. A w-path is a path staring from vertex numbered 1 and ending at any other vertex. The weight along a w-path is the sum of the weights of the vertices on this path.*

What is the minimum length L of the path you need to ensure that the weight along any w-path of length L is at least 79/40.

(a) 6

(b) 7

(c) 8

(d) 9

Problem 10. *You are playing a game on a 2d-coordinate plane. And there is one marker of weight 1 kept at the origin (0,0). In each move you can pick up a marker placed at (x,y), break it up into two equal parts of weights and place one of the part at (x,y+1) and the other at (x+1,y).*
Example - in the beginning with one marker at the origin. you divide the marker into parts of weight $\frac{1}{2}$ *each and place them at (0,1) ans (1,0).*
So now in the plane there are two markers of weight $\frac{1}{2}$*.*

$$g(i,j) = \begin{cases} 1 & \text{if there is a marked placed in (i, j)} \\ 0 & \text{otherwise} \end{cases}$$

What is invariant after every move?

1) The total sum of weight of stones.
2) $\sum_{(i,j)} \frac{g(i,j)}{2^{i+j}}$
3) $\sum_{(i,j)} \frac{g(i,j)}{2^{i}}$

(a) 1 and 2 are invariant.

(b) 1 and 3 are invariant.

(c) 2 and 3 are invariant.

(d) 1, 2 and 3 are invariant.

Subjective Questions

Problem 11. *We have m persons arranged in a circular fashion. Prove that the number of ways of choosing k from among them, such that no two persons chosen are consecutive, is $\frac{m}{k}\binom{m-k-1}{k-1}$.*

Problem 12. *Thales and Carol are playing a game. In the beginning you are given 5 points. In each turn Thales can join any two point yet to be joined with a line and color it with red and in each turn Carol can join any two point yet to be joined with a line and color it with blue. Thales goes first. And they play the game alternatively. Thales wins if at some point of time there is a triangle with all its edges colored red and their vertices are from the 5 point given in the beginning. Can Carol prevent Thales from winning ?*

Problem 13. *Consider the following game. A and B are each given a positive integer. Both of them are told that the numbers are consecutive but neither know the other person's number. For example, if A is told 14, he does not know if B was told 13 or 15. And if B is told 15, he does not know if A was told 14 or 16. The point of the game is to guess the other person's number. The game works as follows:-*

- *A and B cannot communicate with each other, and they are not allowed to plan their strategy either.*
- *The two are in a room where a clock rings every minute.*
- *After the clock rings, either player can call out a guess of the other player's number, or they can stay silent.*
- *The game continues until A or B makes a guess. After the first guess is made, the game ends.*
- *A and B win $1 million each if the guess is correct, and they lose and get nothing if the guess is incorrect.*

How should A and B play this game to have the best chance of winning? Each knows the other person is perfect at logical reasoning.

Problem 14. *A hyper-clock is a collection of k clocks $C_1, C_2, \cdots C_k$. Clock C_i has a dial with numbers from 1 to n_i (arranged in a cyclic manner) along with a hand that is pointing towards one of the numbers.*

To join the council of time lords, you must pass the following test:

- *You are given a hyper-clock where all the clocks have their hands pointing to 1.*
- *You are allowed single step operations of moving one of the clock hands to the right or to the left.*
- *You must execute a sequence of these single step operations, so that every possible configuration of the dials of the hyper-clock are achieved exactly once.*

Show that this can indeed be done.

Example:

1. Suppose there was only one clock in the hyper-clock with say numbers from 1 to 12. Then, the task can be solved simply by rotating the hand one step clockwise every time. 2. Suppose there were three clocks, with each of them having the numbers 1 and 2. Then, you could solve the task by following this sequence: '111', '112', '121', '122', '222', '221', '212', '211'

Problem 15. *Alice has a secret number* $S < 2^N$ *(with* $N \leq 32$ *). Now, you have joined the evil forces with Eve and so you have to help Eve crack Alice's secret number, i.e., find the value of* S*.*

All that Eve has access to is a system with hidden implementation to a special function `xor_with(x)`*. But the system has some restrictions and so you can only make* $N + 1$ *calls to the* `xor_with(x)` *function.*

Everytime `xor_with(x)` *is called with argument* x*, it calculates* $x \oplus S$ *and finally returns the number of 1's in* $x \oplus S$*.*

Outline an algorithm that you can use to obtain the value of S*, or prove that it cannot be done.*