TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

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Mathematics Exam
Category A Practice Paper

October 14, 2018

Rules and Regulations

Marking Scheme

1. The question paper is divided in two parts: Objective and Subjective.
2. Each objective question is worth **1 point** and there is no negative marking.
3. You are not required to show your work for the objective part of the paper.
4. Each subjective problem is worth **20 points**.
5. For getting full credit in the subjective questions you need to give the detailed solutions. However, credit will also be awarded for partially correct solutions.
6. There is no negative marking in the subjective section as well.
7. **The subjective part will be graded only if you score above a certain cut-off (which will be decide later) in the objective section of the paper. However, for the final score, your total score (subjective + objective) will be taken into consideration.**

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.
4. You are not allowed to post/discuss the problems in any online forum within the exam time.

Notation

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ are the set of integers, rationals, positive integers, real numbers and positive real numbers respectively.
2. $\in$ denotes "**belongs to**" , $\forall$ denotes "**for all**", $\exists$ denotes "**there exists**", $\nexists$ denotes "**there does not exist**"

3. $a \mid b$ for any two integers a, b with $a \neq 0$ means that a is a factor of b e.g. 2 is a factor of 6.
4. $a \nmid b$ means that a is **NOT** a factor of b. Example : $2 \nmid 5$ is true
5. "**iff**" denotes "**if and only if**" . For example, "Statement 1 is true" **iff** "Statement 2 is true" means that "Statement 1 is true" implies "Statement 2 is true" and vice-versa.
6. $\sum, \prod$ denote summation and product respectively, for further clarification, visit `https://math.illinoisstate.edu/day/courses/old/305/contentsummationnotation.html`

Objective Questions

For **Problems 1-10**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***one or more than one option*** may be correct. Choosing **ALL** the correct options of a problem will be treated as a **correct** answer, any other answer will be treated as a **wrong** answer. For getting credit, you have to write **ALL** the correct options, i.e. for a problem, if options **(b), (c), (d)** are correct, you mention the concerned problem and ONLY write **(b), (c), (d)**.

For **Problems 11-15**, write **ONLY** the numerical value asked alongside the question number, **NO** justification for the answers is required for the problems in this section.

For **Problems 1-15**, **1 point** will be awarded for correctly answering a problem, **NO** negative marks shall be awarded for wrong answers/unattempted problems .

Problem 1. *How many even positive integers can be made using the digits* $2, 3, 7, 8$ *at most once that are perfect squares?*

(a) *0*

(b) *64*

(c) *32*

(d) ∞

Problem 2. *Let* $f(n) = n^2 - n - 3$. *Choose the correct options.*

(a) *There are infinitely many integers* n *such that* $13 \mid f(n)$.

(b) *There are infinitely many integers* n *such that* $13 \nmid f(n)$.

(c) *There are infinitely many integers* n *such that* $13^2 \nmid f(n)$.

(d) *There are only finitely many integers* n *such that* $13^2 \nmid f(n)$.

Problem 3. *What is the units digit of* 13^{2018} *?*

(A) 1 ***(B)*** 3 ***(C)*** 5 ***(D)*** 6 ***(E)*** 9

Problem 4. *Let $BACD$ be a convex cyclic quadrilateral in which AD bisects $\angle BAC = 60°$, $|AB| = 1$, $|AC| = 2$. Then $|AD|$ is:*

(a) $\sqrt{3}$

(b) $\sqrt{2}$

(c) *1*

(d) $\sqrt{5}$

Problem 5.

Let J be a point in the plane of a given $\triangle ABC$ with $|AB| = 2018$, $|AC| = 2017$, $|BC| = 2016$ such that pedal triangle of J with respect to $\triangle ABC$ is equilateral and is also having the minimal area out of all the equilateral triangles that have their vertices on $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ (one on each of the sides). Then $\dfrac{|JB|}{|JC|}$ is:

(a) $\dfrac{2018^2}{2017^2}$

(b) $\dfrac{2018}{2017}$

(c) $\dfrac{2017}{2018}$

(d) $\dfrac{2016}{2017}$

Problem 6. *$A \subseteq \mathbb{Q}$ is such that $\forall\ z \in A$, we have:*

- $(z+1) \in A$
- $(z-1) \in A$
- $\frac{-1}{z} \in A$ *(if $z \neq 0$)*
- $1 \in A$

Let $S = \{t \mid \frac{p}{q} \in A, gcd(p,q) = 1, t = p^2 + q^2\}$. Find the number of elements of S that are atmost 100. (that is, find $\mid S_{100} \mid$ where $S_{100} = \{x \mid x \in S, x \leq 100\}$)

(a) *23*

(b) *22*

(c) *24*

(d) *21*

Problem 7. *How many even polynomials are there of degree at most* 10 *whose coefficients are single digit positive integers?*
An even polynomial is any polynomial P *such that* $P(x) = P(-x) \forall x$

(a) $1000,000$

(b) $100,000$

(c) $500,000$

(d) 5^{11}

Problem 8. *How many* 3 *digit numbers are there that have an odd number of divisors?*

(a) *23*

(b) *24*

(c) *30*

(d) *31*

Problem 9. *Find the number of polynomials with degree at* $-$ *most 2018 (less than or equal to 2018) such that* $P(x^2) = P(x) \cdot (P(x) + 2x) \ \forall \ x$.

(a) *2019*

(b) *2018*

(c) *1009*

(d) ∞

Problem 10. *Find the number of pairs* $\{a, b\}$ *such that* $a{+}b$ = *2017;* $a, b \in \mathbb{N}$ *and there is no carry over while adding* a *and* b *in base 10.*

(a) *48*

(b) *47*

(c) *49*

(d) *50*

Problem 11. *Given* $\triangle ABC$, P *be foot of* A*-angle bisector and* OH,BC *intersect at* X. *If* $AX \perp PH$, *then find all possible values of* $\angle A$
Note. O *is the circumcentre of* $\triangle ABC$, H *is orthocentre of* $\triangle ABC$.

Problem 12. *Find the least possible value of* $a^4 + b^4 + c^4 + d^4 - 4abcd$ *where* $a, b, c, d \in \mathbb{R}$.

Problem 13. *$A = \{1, 2, \cdots, n\}$, $n \geq 2$. Find a closed form (that is,* ***free*** *of any summation $\sum$ or product symbols $\prod$ in its expression) of the number of ways of choosing $P_1, P_2, \cdots, P_k \subseteq A$ such that $\mid P_1 \cap P_2 \cdots \cap P_k \mid \geq 2$.*

Problem 14. *Find the largest positive integer such that $n - \phi(n) = 10$ where $\phi(n)$ is the Euler-totient function.*

Problem 15. *Find the smallest prime number p such that $p \mid n^2 - n - 2023$ for some integer n.*

Subjective Problems

Problem 1. *Prove that* 2005^2 *can be written in at least* 4 *ways as the sum of 2 perfect (non-zero) squares.*

Problem 2. *Prove that if for a real number* a *,* $a + \frac{1}{a}$ *is integer then* $a^n + \frac{1}{a^n}$ *is also integer where* n *is positive integer.*

Problem 3. *A natural number* n *was alternately divided by* 29*,* 41 *and* 59*. The result was three nonzero remainders, the sum of which equals* n*. Find all such* n*.*

Problem 4. *Given points* A, B *with* $\mid AB \mid = 12m$*, a point* P *is called good if the length of* $A-$*median and* $B-$*altitude are equal in* $\triangle PAB$*. Find the maximum possible distance between any two good points.*

Problem 5. S *is a finite subset of* $\mathbb{R}^2$*. Such that :*

- *No three points of* S *are collinear.*
- *If* $A, B, C \in S$*; then* $A', B', C' \in S$ *where* A', B', C' *are diametrically opposite to* A, B, C *(respectively) with respect to the circumcircle of triangle* ABC*.*

Prove that all points in S *are concyclic.*

Problem 6. *The deputies in a parliament were split into* 10 *fractions. According to regulations, no fraction may consist of less than five people, and no two fractions may have the same number of members. After the vacation, the fractions disintegrated and several new fractions arose instead. Besides, some deputies became independent. It turned out that no two deputies that were in the same fraction before the vacation entered the same fraction after the vacation. Find the smallest possible number of independent deputies after the vacation.*