TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

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Mathematics Exam

Category B Practice Paper

October 14, 2018

Rules and Regulations

Marking Scheme

1. The question paper is divided in two parts. Objective and Subjective.
2. Each objective question is worth **1 point** and there is no negative marking.
3. You are not required to show your work for the objective part of the paper.
4. There are a total of **8** subjective problems, out of which, the best of your **SIX** attempts will be considered for evaluation in the case you score beyond cut-off in the objective section.
5. Each subjective problem is worth **20 points**.
6. For getting full credit in the subjective questions you need to give the detailed solutions. However, credit will also be awarded for partially correct solutions.
7. There is no negative marking in the subjective section as well.
8. **The subjective part will be graded only if you score above a certain cut-off (which will be decide later) in the objective section of the paper. However, for the final score, your total score (subjective + objective) will be taken into consideration.**

Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.
4. You are not allowed to post/discuss the problems in any online forum within the exam time.
5. For any querries regarding the paper, please email `stems.school.math@gmail.com`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `stems.tessellate@gmail.com` .

Notation

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ are the set of integers, rationals, positive integers, real numbers and positive real numbers respectively.
2. $\in$ denotes "**belongs to**" , $\forall$ denotes "**for all**", $\exists$ denotes "**there exists**", $\nexists$ denotes "**there does not exist**"
3. $a \mid b$ for any two integers a, b with $a \neq 0$ means that a is a factor of b e.g. 2 is a factor of 6.
4. $a \nmid b$ means that a is **NOT** a factor of b. Example : $2 \nmid 5$ is true
5. "**iff**" denotes "**if and only if**" . For example, "Statement 1 is true" **iff** "Statement 2 is true" means that "Statement 1 is true" implies "Statement 2 is true" and vice-versa.
6. $\sum, \prod$ denote summation and product respectively, for further clarification, visit `https://math.illinoisstate.edu/day/courses/old/305/contentsummationnotation.html`

Objective Questions

For **Problems 1-10**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***one or more than one option*** may be correct. Choosing **ALL** the correct options of a problem will be treated as a **correct** answer, any other answer will be treated as a **wrong** answer. For getting credit, you have to write **ALL** the correct options, i.e. for a problem, if options **(b), (c), (d)** are correct, you mention the concerned problem and ONLY write **(b), (c), (d)**.

For **Problems 11-16**, write **ONLY** the numerical value asked alongside the question number, **NO** justification for the answers is required for the problems in this section.

For **Problems 1-16**, **1 point** will be awarded for correctly answering a problem, **NO** negative marks shall be awarded for wrong answers/unattempted problems .

Problem 1. *We call an integer m* ***good*** *if the set $S(m) = \{1, 2, 3, \cdots 2m\}$ can be split into m disjoint sets each containing two elements such that the sum of the two numbers in each set is a prime. What is the probability of choosing a* ***good*** *number from the set of natural numbers?*

(a) $\frac{1}{2}$

(b) $\frac{1}{e}$

(c) *0*

(d) *1*

Problem 2. *Let $f(n) = n^2 - n - 2018$. Choose the correct options.*

(a) *There are infinitely many integers n such that $17 \mid f(n)$.*

(b) *There are infinitely many integers n such that $17 \nmid f(n)$.*

(c) *There are infinitely many integers n such that $17^2 \nmid f(n)$.*

(d) *There are only finitely many integers n such that $17^2 \nmid f(n)$.*

Problem 3. *Alice and Bob are playing a game. At the beginning of the game, there is a cubic polynomial with integral coefficients written on the blackboard, which we denote as the* ***starting polynomial****. The two players take turns one by one. In each turn, the player*

either *chooses any natural number* n *and replaces the existing cubic polynomial* $f(x)$ *on the blackboard by any one of* $f(n+x)$, $f(nx)$, $f(x)+n$ ***or*** *just changes the sign of the coefficients of* x^2*, i.e. if the existing polynomial is* $a_0 + a_1x + a_2x^2 + a_3x^3$*, then he can replace it by* $a_0 + a_1x - a_2x^2 + a_3x^3$*. Alice takes the first turn. Bob wins if, after finitely many moves, the cubic polynomial on the blackboard has all coefficients (upto* x^3*) non-zero and equal. For which of the* ***starting polynomials*** *can Alice ensure that Bob does not win in finite number of moves?*

(a) $x^3 + 4x^2 + 4x + 1$

(b) $x^3 + 6x^2 + 7x$

(c) $x^3 + 3$

(d) *none of the above*

Problem 4. *Let* $BACD$ *be a convex cyclic quadrilateral in which* AD *bisects* $\angle BAC = 60°$, $|AB| = 1$, $|AC| = 2$*. Then* $|AD|$ *is:*

(a) $\sqrt{3}$

(b) $\sqrt{2}$

(c) *1*

(d) $\sqrt{5}$

Problem 5. *Evaluate* $\lim_{x \to 1} x^{\frac{x}{\sin(1-x)}}$

(a) $\frac{1}{e}$

(b) $\frac{1}{e^2}$

(c) 0

(d) 1

Problem 6. $A \subseteq \mathbb{Q}$ *is such that* $\forall\ z \in A$*, we have:*

- $(z+1) \in A$
- $(z-1) \in A$
- $\frac{-1}{z} \in A$ *(if* $z \neq 0$*)*
- $1 \in A$

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Let $S = \{t \mid \frac{p}{q} \in A, gcd(p,q) = 1, t = p^2 + q^2\}$*. Find the number of elements of* S *that are atmost 100. (that is, find* $\mid S_{100} \mid$ *where* $S_{100} = \{x \mid x \in S, x \leq 100\}$*)*

(a) *23*

(b) *22*

(c) *24*

(d) *21*

Problem 7. *Evaluate* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$

(a) $2 \log_e 2$

(b) $1 + \sin 2$

(c) 2

(d) 1.5

Problem 8. *Let* f *be a formal power series with every coefficient either 0 or 1; with* $f\left(\frac{2}{3}\right) = \frac{2017}{2^{2018}}$*. What is the period of decimal expansion of* $f\left(\frac{1}{2}\right)$.

(a) *2017*

(b) *2018*

(c) 2^{2018}

(d) ∞ *(that is,* $f\left(\frac{1}{2}\right)$ *is irrational)*

Problem 9. *Let* $f(x) = \sin(\sin(x))$*. Evaluate*

$$\lim_{h \to 0} \frac{f(x+h) - f(h)}{x}$$

at $x = \pi$.

(a) *0*

(b) *-1*

(c) *1*

(d) *does not exist*

Problem 10. *Find the number of pairs $\{a, b\}$ such that $a+b = 2017$; $a, b \in \mathbb{N}$ and there is no carry over while adding a and b in base 10.*

(a) *48*

(b) *47*

(c) *49*

(d) *50*

Problem 11. *Given $\triangle ABC$, P be foot of A-angle bisector and OH,BC intersect at X. If $AX \perp PH$, then find all possible values of $\angle A$*
Note. *O is the circumcentre of $\triangle ABC$, H is orthocentre of $\triangle ABC$.*

Problem 12. *$A = \{1, 2, \cdots, n\}$, $n \geq 2$. Find a closed form (that is, **free** of any summation $\sum$ or product symbols $\prod$ in its expression) of the number of ways of choosing $P_1, P_2, \cdots, P_k \subseteq A$ such that $\mid P_1 \cap P_2 \cdots \cap P_k \mid \geq 2$.*

Problem 13. *Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.*

Problem 14. *A triangle has sides with lengths a, b, c such that*

$$a^2 + b^2 = 5c^2$$

Calculate the angle between the medians to the sides of lengths a and b.

Problem 15. *Let $f(x) = \underbrace{\sin(\sin(\sin(\sin(\sin(x)))))}_{5 \quad times}$, and suppose that the number α satisfies the equation $\alpha = \sin\alpha$. Express $f'(\alpha)$ as a polynomial in α, also compute the number of polynomials with coefficients in non-negative integers of value at most 2, of degree at most 10, each of which has α as one of its roots.*

Subjective Problems

The following are the subjective problems, the **BEST** of your **SIX** attempts shall be evaluated and considered for your score in the subjective section, however, you are encouraged to attempt more than six problems.

Problem 1. *Given a prime* $p \equiv 3 \pmod 4$. *Prove that for all given integers* a, b *with* $p \nmid gcd(a, b)$, $\exists\ c, d \in \mathbb{Z}$ *so that* $p \mid ac - bd - 1$ *and* $p \mid ad + bc$.

Problem 2. *Let* ABC *be an acute-angled triangle with circumcenter* O. *Let* P *and* Q *be the reflections of points* B *and* C *in the lines* AC *and* AB *respectively. Prove that the lines* AO, BQ, *and* CP *concur.*

Problem 3. S *is a finite subset of* $\mathbb{R}^2$. *Such that :*
- *No three points of* S *are collinear.*
- *If* $A, B, C \in S$; *then* $A', B', C' \in S$ *where* A', B', C' *are diametrically opposite to* A, B, C *(respectively) with respect to the circumcircle of triangle* ABC.

Prove that all points in S *are concyclic.*

Problem 4. *Prove that given any polynomial* $P(X)$ *with integral coefficients,* $\nexists M \in \mathbb{N}$ *such that* $P(n)$ *is square-free* $\forall\ n \geq M$.

Definition. $T \in \mathbb{N}$ *is said to be square-free if* $p^2 \nmid T\ \forall$ *primes* p.

Problem 5. *Let* $n = p + 1$ *where* p *is an odd prime. A group of* n *scientists attend a seminar. Some of them shake hands with each other. Two scientists* A, B *are said to be connected if either of the following are true:*

1. A *and* B *shake hands.*
2. *There are scientists* $S_1, \cdots, S_k$ *for some* $k \geq 1$ *such that* A *shakes hands with* S_1, S_i *shakes hands with* $S_{i+1}\ \forall\ 1 \leq i \leq (k-1)$ *and* S_k *shakes hands with* B.

Let t_n *be the number of such scenarios in which scientists have shook hands in such a way that any two scientists attending the seminar are connected. Prove that* $t_n^2 \equiv 1 \pmod p$.

Definition. *Two scenarios are said to be different if there exists a pair of scientists who shook hands in one scenario but not in the other.*

Problem 6. *Let* $a_{n+1} = a_n + \frac{1}{{a_n}^{2018}}$ *and* $a_1 = 1$.
Show that $\exists C > 0$ *such that the sum* $\sum_{n=1}^{M} \frac{1}{na_n} < C \quad \forall M \in \mathbb{N}$.

Problem 7. *Let* $A \otimes B := \{a + b \mid a \in A, b \in B, a \neq b\}$ *for any two sets* $A, B \subset \mathbb{N}$. *Prove that there exists a unique partition of* $\mathbb{N}$ *into* U, V *such that* $U \otimes U, V \otimes V$ *contains none of the prime numbers.*

Problem 8. *Let* $0 < \alpha < \beta$. *Prove that* $\int_{\alpha}^{\beta} (x^2 + 1)e^{-x^2} dx \geq e^{-\alpha^2} - e^{-\beta^2}$.