TESSELLATE PRESENTS

# STEMS

Scholastic Test of Excellence in Mathematical Sciences

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# Mathematics Exam
# Category C Practice Paper

October 14, 2018

# Rules and Regulations

## Marking Scheme

1. The question paper is divided in two parts: Objective and Subjective.
2. Each objective question is worth **1 point** and there is no negative marking.
3. You are not required to show your work for the objective part of the paper.
4. Each subjective problem is worth **20 points**.
5. For getting full credit in the subjective questions you need to give the detailed solutions. However, credit will also be awarded for partially correct solutions.
6. There is no negative marking in the subjective section as well.
7. **The subjective part will be graded only if you score above a certain cut-off (which will be decide later) in the objective section of the paper. However, for the final score, your total score (subjective + objective) will be taken into consideration.**

## Miscellaneous

1. Any form of plagiarism will lead to disqualification
2. For solving the problems, you are allowed to use the Internet and books as resources.
3. Write the answer clearly, in a legible way. Write formal proofs wherever necessary. Be clear with your reasoning.
4. You are not allowed to post/discuss the problems in any online forum within the exam time.
5. For any querries regarding the paper, please email `stems.school.math@gmail.com`. We shall respond to you as soon as possible. Note that the email mentioned above is only for clarifying doubts related to question paper. Technical queries must be emailed at `stems.tessellate@gmail.com` .

# Notation

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ are the set of integers, rationals, positive integers, real numbers and positive real numbers respectively.

2. $\in$ denotes "**belongs to**" , $\forall$ denotes "**for all**", $\exists$ denotes "**there exists**", $\nexists$ denotes "**there does not exist**"

3. $a \mid b$ for any two integers $a, b$ with $a \neq 0$ means that $a$ is a factor of $b$ e.g. 2 is a factor of 6.

4. $a \nmid b$ means that $a$ is **NOT** a factor of $b$. Example : $2 \nmid 5$ is true

5. "**iff**" denotes "**if and only if**" . For example, "Statement 1 is true" **iff** "Statement 2 is true" means that "Statement 1 is true" implies "Statement 2 is true" and vice-versa.

6. $\sum, \prod$ denote summation and product respectively, for further clarification, visit `https://math.illinoisstate.edu/day/courses/old/305/contentsummationnotation.html`

# Objective Questions

For **Problems 1-10**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***one or more than one option*** may be correct. Choosing **ALL** the correct options of a problem will be treated as a **correct** answer, any other answer will be treated as a **wrong** answer. For getting credit, you have to write **ALL** the correct options, i.e. for a problem, if options **(b), (c), (d)** are correct, you mention the concerned problem number and ONLY write **(b), (c), (d)**.

For **Problems 1-10**, **1 point** will be awarded for correctly answering a problem. However, nothing will be awarded or deducted for wrong answers/unattempted problems.

**Problem 1.** *Consider the set of functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that for every regular $n$-gon $A_1A_2\ldots A_n$ we have $f(A_1) + f(A_2) + \cdots + f(A_n) = 0$.*
*Notice that this is a subspace of the real vector space of all functions $g : \mathbb{R}^2 \to \mathbb{R}$. What is the dimension of this subspace?*

(a) *0*

(b) *1*

(c) *2*

(d) $\infty$

**Problem 2.** *Let $S$ be a proper subgroup of the group $(\mathbb{R}, +)$. Suppose for all differentiable functions $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that $f^{-1}(S)$ is dense inside $\mathbb{R}$, the derivative of $f$ vanishes at some point in $\mathbb{R}$. Which of the following options are correct?*

(a) *$S$ can be the trivial subgroup.*

(b) *$S$ is closed inside $\mathbb{R}$.*

(c) *$S$ can be a dense subset of $\mathbb{R}$.*

(d) *none of the above*

**Problem 3.** *Consider the multiplicative group $A = \{z \in \mathbb{C} | z^{2018^k} = 1, 0 < k \in \mathbb{Z}\}$ of all the roots of unity of degree $2018^k$ for all positive integers $k$.*

*Find the number of homomorphisms $f : A \to A$ that satisfy $f(f(x)) = f(x)$ for all elements $x \in A$.*

(a) *4*

(b) *8*

(c) *1*

(d) $\infty$

**Problem 4.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $P_n(x) = \int_0^{x^2} f_n(t)dt$, $\forall n \in \mathbb{N}$.*
*Which of the following options are correct?*

(a) *If $\{P_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function $P$, then $\{f_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function $f$.*

(b) *If $\{f_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function $f$, then $\{P_n\}_{n\in\mathbb{Z}}$ converges uniformly over $\mathbb{R}$ to a function $P$.*

(c) *If $f_1$ is uniformly continuous in $\mathbb{R}$, then $P_1$ is uniformly continuous in $\mathbb{R}$.*

(d) *none of the above*

**Problem 5.** *Let $G$ be a graph with $V$ as the set of its vertices and $E$ as the set of its edges. Suppose $f$ is a function from $E$ to the set of integers. We call function $g : V \to \mathbb{Z}$ $f$-**good** if for any two neighbouring vertices $v_1, v_2 \in V$, $g(v_1) + g(v_2) = f(v_1, v_2)$ where $(v_1, v_2)$ is the edge between $v_1$ and $v_2$. Suppose for all functions $f_1 : E \to \mathbb{Z}$, there exist infinitely many $f_1$-**good** functions $g_1 : V \to \mathbb{Z}$, if at all there exists one. Then, $G$ can have a cycle of length*

(a) *3*

(b) *4*

(c) *5*

(d) *6*

**Problem 6.** *$\mathfrak{R}$ be a non-commutative ring. With or without unity.*
*Choose the correct alternatives.*

(a) *There might exist $a, b \in \mathfrak{R}$ such that $ab$ is a zero divisor, but $ba$ is not.*

(b) *$ab$ is a zero divisor $\iff$ $ba$ is a zero divisor $\forall a, b \in \mathfrak{R}$.*

(c) $ab = 0 \iff 0 \in \{a, b\} \forall a, b \in \mathfrak{R}$

(d) *none of the above*

**Problem 7.** *Let $G$ be a non-abelian group with no proper, non-trivial, normal subgroups. How many proper, non-trivial normal subgroups does $G \times G$ have?*

(a) *2*

(b) *3*

(c) *4*

(d) *depends on the group $G$ chosen*

**Problem 8.** *$\mathfrak{R}$ be a non-commutative ring.*
*Choose the correct alternatives.*

(a) *$a \in \mathfrak{R}$ might have exactly 2018 right inverses for some $a \in \mathfrak{R}$.*

(b) *$a \in \mathfrak{R}$ has a unique right inverse for any right unit $a \in \mathfrak{R}$ if $\mathfrak{R}$ is finite.*

(c) *$\mathfrak{R}$ is infinite.*

(d) *none of the above*

**Problem 9.** *Let $f(x,y) = \sum_{i=1}^{n} a_i x^i y^{i-1}$ where $a_i \in \mathbb{R}$ with $a_n \neq 0$. We are given that $f$ satisfies the property : For all $y_0 \in (-1,4)$*

$$\sum_{i=1}^{n-1} i a_i x_0^{i-1} y_0^{i-1} = 0 \implies \text{ for all } x \in U(x_0)\ ,\ f(x,y_0) \text{ is constant}$$

*where $U(x_0)$ is a neighborhood around $x_0$*
*Let $g_y(x) = f(x,y)$ for all $y \in (-1,4)$. Then which of the following is true:*

(a) *$g_y$ is surjective onto $\mathbb{R}$ for all $y \in (2,3)$*

(b) *$g_y$ is surjective onto $\mathbb{R}$ for all $y \in (1,4)$*

(c) *$g_y$ is surjective onto $\mathbb{R}$ for all $y \in (-1,1)$*

(d) *None of the above*

**Problem 10.** *For $n \times n$ matrices $A$ and $B$ with real entries we define a function $f_{A,B} : \mathbb{R} \to \mathbb{R}$, $f_{AB}(x) = det(xA + B)$. Choose the correct options.*

(a) *if there exists matrices $A, B$ with real entries such that the function $f_{AB}(x)$ is bounded, then $det(A) = 0$.*

(b) *For all matrices $A, B$ with real entries, the function $f_{AB}(x)$ vanishes at a point.*

(c) *There exists matrices $A, B$ with real entries such that the function $f_{AB}(x)$ vanishes at a point.*

(d) *None of the above*

Each problem in **Problems 11-15** has an integer valued answer. For getting credit, you have to correctly write the integer, i.e. if the answer to a problem is **5**, you write the concerned problem number and write **5** beside it. Each problem is worth **1 point**. However, **NO** points will be deducted or awarded for wrong answers/unattempted problems.

**Problem 11.** *Find the number of injective group homomorphisms from $S_3$ to $S_5$ ($S_n$ denotes the permutation group on a set with $n$ elements).*

**Problem 12.** *We call an integer $n$ **basy** if for all real valued $2018 \times 2018$ matrices $A$ such that $A^2 = -nI$ (where $I$ is the identity matrix of size $2018 \times 2018$), there exist vectors $\{v_1, v_2, \cdots, v_{1009}\}$, $v_i \in \mathbb{R}^{2018}$ $\forall i$, such that the set $\{v_1, v_2, \cdots, v_{1009}, A(v_1), A(v_2), \cdots A(v_{1009})\}$ forms a basis for $\mathbb{R}^{2018}$. Then how many **basy** integers are there in the set $\{1, 2, \cdots, 10\}$?*

**Problem 13.** *Calculate the number of homomorphisms from $A_5$ to $D_{60}$ where:*

*$D_{60}$ is the dihedral group. For more details, see* `https://en.wikipedia.org/wiki/Dihedral_group`

*$A_5$ is the alternating group with 5 elements. For more details, see* `https://en.wikipedia.org/wiki/Alternating_group`

**Problem 14.** *Let $A_1, A_2, \cdots, A_9$ be 9 players playing a round-robin tournmaent ( each player plays every other player exactly once). Every game results in either a win or a loss. At the end of the tournament, it is seen that the total number of wins of $A_1, A_2, \cdots, A_9$ are respectively $4, 3, 6, 4, 5, 5, 1, 4, 4$. Find the number of tuples $(A_p, A_q, A_r)$, $p, q, r \in \{1, \cdots, 9\}$ such that player $A_p$ has beaten player $A_q$, player $A_q$ has beaten player $A_r$, player $A_r$ has beaten player $A_p$.*

**Problem 15.** *Let $\lambda : \mathbb{N} \mapsto \mathbb{R}$ be the function*

$$\lambda(n) = \sum_{k=1}^{\infty} \frac{\phi(\gcd(k, n))}{\operatorname{lcm}(k, n)^2}.$$

*It is well-known that* $\lambda(1) = \frac{\pi^2}{6}$. *What is the smallest positive integer* $m$ *such that* $m \cdot f(2018)$ *is the square of a rational multiple of* $\pi$*?*
*where:*

$\phi(m)$ *is the Euler's Totitent Function. For more details, visit* ***https://en.wikipedia.org/wiki/Euler%27s_totient_function***

# Subjective Questions

**Problem 1.** *Given any sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.*
***Statement 0*** *: $\sum_{n\in\mathbb{N}} \mid a_n \mid$ is convergent.*
***Statement 1*** *: For all but finitely many bijections $f : \mathbb{N} \to \mathbb{N}$, $\sum_{n\in\mathbb{N}} a_{f(n)}$ is convergent.*
***Statement 2*** *: For infinitely many bijections $f : \mathbb{N} \to \mathbb{N}$, $\sum_{n\in\mathbb{N}} a_{f(n)}$ is convergent.*

*Prove or disprove the following :*

1. ***Statement 1*** $\implies$ ***Statement 0***
2. ***Statement 2*** $\implies$ ***Statement 0***.

**Problem 2.** *$G$ is a group. Prove that the following are equivalent: 1. All subgroups of $G$ are normal. 2. For all $a, b \in G$ there is an integer $m$ such that $(ab)^m = ba$.*

**Problem 3.** *Find all natural numbers such that*

$$n\sigma(n) \equiv 2 \pmod{\phi(n)}$$

*where:*

*$\phi(n)$ is the Euler's Totitent Function. For more details, visit* ***https://en.wikipedia.org/wiki/Euler%27s_totient_function***

*$\sigma(n)$ is the sum of positive divisors of $n$*

**Problem 4.** *Prove or disprove the following:*
*There exists a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $[0,1] \times [0,1] \subset f(\mathbb{Q}^2)$ where $f(\mathbb{Q}^2) = \{x \mid x = f(y) \quad \text{for some} \quad y \in \mathbb{Q}^2\}$. Can one say the same about $\mathbb{Q} \times \mathbb{R}$ instead of $\mathbb{Q}^2$?*

**Problem 5.** *Let $R$ be an integral domain. Suppose $R$ has finitely many ideals. Prove that $R$ is a field.*
*You may find this* ***https://en.wikipedia.org/wiki/Integral_domain*** *helpful.*

**Problem 6.** *Let $G$ be a connected graph. A subset of its edges is called bunny if each vertex has even degree in the subgraph generated by the elements of that subset. Prove that the number of bunny subsets is a function of the number of edges and the number of vertices of $G$.*

*(To be precise, prove that if $G_1$ and $G_2$ are two connected graphs with same number of edges and vertices, then they have the same number of bunny subsets.)*